ON THE RELATIONSHIP BETWEEN ZERO ENTROPY AND QUASI-DISCRETE SPECTRUM FOR AFFINE TRANSFORMATIONS

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Introduction. If \( T \) is a measure-preserving transformation of a Lebesgue space \((M, \mathcal{B}, m)\) let \( \pi(T) \) be the maximum partition such that \( T^*\pi(T) \) has zero entropy. In §2 we prove a result, which was stated without proof in [12], concerning the behaviour of the partitions \( \pi(T_{t_n}) \) associated with an increasing sequence of invariant measurable partitions \( \{t_n\} \).

If \( T \) is an ergodic affine transformation of a compact connected metric abelian group \( G \) let \( \eta(T) \) be the maximum partition such that \( T^*\eta(T) \) has quasi-discrete spectrum. In §3 we prove, using the result of §2 and a method introduced by Rohlin in [8], that \( \eta(T) = \pi(T) \).

This result was first obtained by Parry [6] who also proved that the maximum partition \( \gamma(T) \) such that \( T^*\gamma(T) \) has the distal property is also \( \pi(T) \).

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1. Preliminaries.

1.1. The partition \( \pi(T) \). For the definitions, notations and theory of Lebesgue spaces and entropy cf. [9], [10], [11].

Let \( Z \) be the set of all measurable partitions of the Lebesgue space \((M, \mathcal{B}, m)\) which have finite entropy and let \( T \) be a (possibly many to one) measure-preserving transformation of \((M, \mathcal{B}, m)\).

The maximum partition \( \pi(T) \) such that \( h(T^*\pi(T)) = 0 \) is given by \( \pi(T) = \bigvee_{n \in Z} \left( \bigwedge_{k=1}^{\infty} T^{-n} \xi_k \right) \) where \( \xi_k = \bigvee_{k=1}^{\infty} T^{-k} \xi \) [11].

We have \( T^{-1}\pi(T) = \pi(T) \), \( \pi(T^n) = \pi(T) \) \( n > 0 \), and if \( T \) is one to one \( \pi(T^{-1}) = \pi(T) \).

If \( \xi \in Z \) and \( \xi \leq \pi(T) \) then \( \xi \leq \xi^{-} \).

\( T \) is said to have completely positive entropy if \( \pi(T) = \nu \), the trivial partition. If \( T \) and \( T' \) are measure-preserving transformations of the Lebesgue spaces \((M, \mathcal{B}, m)\) and \((M', \mathcal{B}', m')\) respectively and if there exists a measure-preserving transformation \( \phi \) of \((M, \mathcal{B}, m)\) onto \((M', \mathcal{B}', m')\) such that \( \phi T = T' \phi \), then \( T' \) is said to be a factor transformation of \( T \) and \( h(T') \leq h(T) \) [11].

1.2. Affine transformations and group partitions. Let \( G \) be a compact connected metric abelian group equipped with completed Haar measure. \( G \) is a Lebesgue space [10]. All group operations are written
additively. Since $G$ is connected its character group $\Gamma$ has no elements of finite order [7]. Endomorphisms of $G$ and their duals will be denoted by the same symbol.

An affine transformation of $G$ is a transformation of the form $T(g) = a + A(g)$, $g \in G$, where $a \in G$ and $A$ is a continuous endomorphism of $G$ onto $G$. Affine transformations preserve Haar measure. $T(g) = a + A(g)$, $g \in G$, is ergodic if and only if the smallest closed subgroup containing $a$ and $(A-I)G$ is $G$ and the only finite orbits under $A$ in the character group are fixed elements [3]. Since $G$ is connected, if $T$ is ergodic $T$ is totally ergodic (i.e., $T^n$ is ergodic for all $n > 0$).

If $BG = G$, where $B = A-I$, then $T(b+g) = b + A(g)$, $g \in G$, where $B(b) = -a$ and $T$ has completely positive entropy. (This follows from the fact that $A$ has completely positive entropy [8].)

Let $H$ be a subgroup of $G$. (All subgroups are assumed closed.) The measurable partition of $G$ into cosets of $H$ will be denoted by $\xi(H)$ [8]. Such a partition is called a group partition. If $H_1 \subseteq H_2$ then $\xi(H_1) \supseteq \xi(H_2)$, $\xi(G) = \nu$, and $\xi(0) = e$, the partition of $G$ into points. If $H_1 \subseteq H_2 \subseteq \cdots$ then $\xi(\bigvee H_n) = \bigwedge \xi(H_n)$ where $\bigvee H_n$ denotes the smallest closed subgroup containing all the $H_n$, and if $H_1 \supseteq H_2 \supseteq \cdots$ then $\xi(\bigwedge H_n) = \bigvee \xi(H_n)$ where $\bigwedge H_n$ is the intersection of the groups $H_n$.

The partition $\alpha(T) = \bigwedge_{n=0}^{\infty} T^{-n}e$ is the finest partition of $G$ which satisfies $T^{-1}\alpha(T) = \alpha(T)$. In particular $\pi(T) \leq \alpha(T)$. $\alpha(T)$ is a group partition since

$$\alpha(T) = \alpha(A) = \bigwedge_{n=0}^{\infty} A^{-n}e = \xi\left(\bigvee_{n=1}^{\infty} \text{kernel } A^n\right).$$

1.3. **Affine transformations with quasi-discrete spectrum.** The definition of a totally ergodic measure-preserving transformation with quasi-discrete spectrum is given in [1] and the definition of a homeomorphism with quasi-discrete spectrum is in [2]. Let $T(g) = a + A(g)$ be an ergodic affine transformation of a compact connected metric abelian group $G$ with character group $\Gamma$. If $\Gamma' = \{ \gamma \in \Gamma \mid B^n\gamma = 0 \}$ where $B = A - I$, then the group of quasi-eigenfunctions of order $n$ is $K \times \Gamma'$, where $K$ is the circle group. $T$ has quasi-discrete spectrum if and only if $\bigcup_{n=1}^{\infty} \Gamma' = \Gamma$ [4].

The maximum partition $\eta(T)$ such that $T_{\eta(T)}$ has quasi-discrete spectrum is the partition of $G$ into cosets of $\text{ann}(\bigcup_{n=1}^{\infty} \Gamma')$ where $\text{ann}(\bigcup_{n=1}^{\infty} \Gamma')$ is the annihilator of $\bigcup_{n=1}^{\infty} \Gamma'$ i.e. $\eta(T) = \xi(\text{ann}(\bigcup_{n=1}^{\infty} \Gamma'))$ [4].

Abramov [1] has proved that every totally ergodic measure-preserving transformation of a Lebesgue space with quasi-discrete
spectrum has zero entropy. Therefore for $T(g) = a + A(g)$, $g \in G$, we have $\eta(T) \leq \pi(T)$.

2. A limit theorem for $\pi(T)$. If $\zeta$ is a measurable partition of the Lebesgue space $(M, \mathcal{B}, m)$ let $H_\zeta$ denote the projection of $M$ onto $M/\zeta$, i.e. $H_\zeta$ maps a point of $M$ onto the element of $\zeta$ to which it belongs. If $\zeta$ is a measurable partition of $M$ such that $T^{-1}\zeta \leq \zeta$ then let $\pi'(T_\zeta)$ denote $H_\zeta^{-1}\pi(T_\zeta)$.

**Theorem 1.** Let $T$ be a measure-preserving transformation (possibly many to one) of a Lebesgue space $(M, \mathcal{B}, m)$ and let $\{\zeta_n\}$ be a sequence of measurable partitions of $M$ such that $\zeta_1 \leq \zeta_2 \leq \cdots$, $\bigvee_n \zeta_n = \epsilon$, and $T^{-1}\zeta_n \leq \zeta_n$ for all $n$. Then $\pi'(T_{\zeta_1}) \leq \pi'(T_{\zeta_2}) \leq \cdots$ and $\bigvee_n \pi'(T_{\zeta_n}) = \pi(T)$.

**Proof.** Since $\pi'(T_{\zeta_n}) = \pi(T) \wedge \zeta_n$ we have $\pi'(T_{\zeta_1}) \leq \pi'(T_{\zeta_2}) \leq \cdots$. Let $\pi_1(T) = \bigvee_n \pi'(T_{\zeta_n})$.

Since $\pi'(T_{\zeta_n}) \leq \pi(T)$ for all $n$, we have $\pi_1(T) \leq \pi(T)$. It remains to show that $\pi_1(T) \geq \pi(T)$. Let $\xi, \eta \in \pi(T)$ and $\zeta \in \mathcal{Z}$. Then since $\zeta \wedge \xi^- = \xi^-$ we have

$$H(\eta/\xi^-) \geq H(\eta/\xi^- \vee \xi) \geq H(\eta/\xi^- \vee \xi^-) = H(\eta/\xi^- \vee \xi^-) = H((\eta \vee \xi)/(\xi^- \vee \xi^-)) = h(T, \eta \vee \xi) \geq h(T, \eta) = H(\eta/\xi^-),$$

i.e.

$$H(\eta/\xi^- \vee \xi) = H(\eta/\xi^-).$$

But

$$H(\xi/\eta^-) + H(\eta/\xi \vee \eta^-) = H(\xi \vee \eta/\eta^-) = H(\eta/\eta^-) + H(\xi/\eta \vee \eta^-)$$

and therefore

$$H(\xi/\eta^-) = H(\xi/\eta \vee \eta^-).$$

Consequently $H(\xi/T^{-k}\eta^-) = H(\xi/\eta \vee \eta^-)$ for all $k > 0$, and if $\eta^- = \bigwedge_{k=1}^\infty T^{-k}\eta^-$ we have [9]

$$H(\xi/\eta^-) = \lim_{k \to \infty} H(\xi/T^{-k}\eta^-) = H(\xi/\eta \vee \eta^-).$$

Let $\xi, \eta \in \pi(T)$ be fixed. Given $\delta > 0$, $\exists \eta \in \mathcal{Z}$, $\eta \leq \zeta_n$ for some $n$, such that $H(\xi/\eta^-) < \delta$.

Since $\eta^- = \bigwedge_{k=1}^\infty T^{-k}\eta^- \leq \pi'(T_{\zeta_n}) \leq \pi_1(T)$, we have $H(\xi/\pi_1(T)) \leq H(\xi/\eta^-) = H(\xi/\eta \vee \eta^-) \leq H(\xi/\eta^-) < \delta$. Therefore $H(\xi/\pi_1(T)) = 0$ and $\xi \leq \pi_1(T)$. Since this is true for all $\xi \in \mathcal{Z}$, $\xi \leq \pi(T)$, we have $\pi(T) \leq \pi_1(T)$.

**Corollary.** If $T$ is a measure-preserving transformation of a Lebesgue space $(M, \mathcal{B}, m)$ and if $\{\zeta_n\}$ is a sequence of measurable parti-
tions of $M$ such that $\xi_1 \leq \xi_2 \leq \cdots$, $V_n \xi_n = \xi$, and $T^{-1} \xi_n \leq \xi_n$ for all $n$, then $V_n \pi'(T_{T_n}) = \pi'(T_{T_i})$.

**Proof.** Apply Theorem 1 to the transformation $T_t$ on $M/\xi$.

### 3. Zero entropy and quasi-discrete spectrum for affine transformations.

**Theorem 2.** Let $G$ be a compact connected metric abelian group and let $T(g) = a + A(g)$, $g \in G$, be an ergodic affine transformation of $G$. The maximum partition $\pi(T)$ such that $T_{\pi(T)}$ has zero entropy and the maximum partition $\eta(T)$ such that $T_{\eta(T)}$ has quasi-discrete spectrum coincide.

**Proof.** We first show that $\pi(T) = \eta(T)$ if, and only if, $\pi(T_{\alpha(T)}) = \eta(T_{\alpha(T)})$. From §1.2 $\alpha(T) = \xi(F)$ for some subgroup $F$ of $G$ satisfying $AF = F$. Therefore $T_{\alpha(T)}$ is the one to one ergodic affine transformation $T_{\alpha(T)}(g') = a' + A'(g')$, $g' \in G/F$, where $A'$ is the automorphism induced by $A$ in $G/F$ and $a'$ is the coset of $F$ containing $a$. If $H_{\alpha(T)}$ is the projection map of $G$ onto $G/F$ then $H^{-1}_{\alpha(T)} \pi(T_{\alpha(T)}) = \alpha(T) \wedge \pi(T)$ and $H^{-1}_{\alpha(T)} \eta(T_{\alpha(T)}) = \alpha(T) \wedge \eta(T)$. Since $\eta(T) \leq \pi(T) \leq \alpha(T)$ we conclude that $\pi(T) = \eta(T)$ if and only if $\pi(T_{\alpha(T)}) = \eta(T_{\alpha(T)})$.

We divide the proof of the theorem into three parts.

(i) Suppose that $G$ is a finite-dimensional torus. We can suppose, without loss of generality, that $T$ and $A$ are one to one. This follows because $T_{\alpha(T)}$ is a one to one ergodic affine transformation of the finite-dimensional torus $G/F$ and $\pi(T) = \eta(T)$ if and only if $\pi(T_{\alpha(T)}) = \eta(T_{\alpha(T)})$.

The set of periodic points of an automorphism of a finite-dimensional torus is dense i.e. if $A_n = \{g \in G \mid A^ng = g\}$, $n > 0$, then $\bigcup_{n=1}^{\infty} A_n$ is dense in $G$ [8]. If $\phi(g) = g_1 + g$ where $g_1 \in A_n$, then $\phi T^n = T^n \phi$ and hence $\phi^{-1} \pi(T^n) = \pi(T^n)$. Since $\pi(T^n) = \pi(T)$, $n > 0$, $\pi(T)$ is completely invariant under a dense set of translations and is therefore a group partition $\pi(T) = \xi(N)$ [8].

$G_1 = G/N$ is a finite-dimensional torus with character group $\Gamma_1 = \text{ann}(N)$, and $T_{\pi(T)}$ is the ergodic affine transformation $T_1(g) = a_1 + A_1(g)$, $g \in G_1$, where $A_1$ is the automorphism of $G_1$ induced by $A$ and $a_1$ is the coset of $N$ containing $a$.

$\{B_1^m G_1, m \geq 0\}$, where $B_1 = A_1 - I$, is a decreasing sequence of subgroups of $G_1$ and therefore $\{\text{ann}(B_1^m G_1), m \geq 0\}$ is an increasing sequence of subgroups of $\Gamma_1$. Since $\Gamma_1$ is a finitely generated free abelian group $\text{ann}(B_1^n G_1) = \text{ann}(B_1^{n+1} G_1)$ for some $n$ and $B_1^n G_1 = B_1^{n+1} G_1$.

Define an affine transformation $S$ on the compact connected metric abelian group $B_1^n G_1$ by $S(B_1^n g) = B_1^n a + A_1(B_1^n g)$, $g \in G_1$. Since $A_1$ and $B_1$ commute we have $SB_1^n = B_1^n T_1$. Since $B_1(B_1^n G_1) = B_1^n G_1$, $S$ has com-
pletely positive entropy (see §1.2). But \( h(T_1) = h(T_{x(T)}) = 0 \) and therefore \( S \), being a factor transformation of \( T_1 \), has zero entropy (see §1.1).

Therefore \( B_1^n G_1 = \{ 0 \} \), and

\[
\Gamma'_n = \text{ann} (B^n_1 G_1) = \{ \gamma \in \Gamma_1 \mid \gamma(B^n_1 g) = 0, \text{ all } g \in G_1 \} = \Gamma_1,
\]

i.e. every element of \( \Gamma_1 \) is a quasi-eigenfunction of order \( n \) (see §1.3), and therefore \( T_1 \) has quasi-discrete spectrum. Therefore \( \eta(T) \geq \pi(T) \) and hence \( \eta(T) = \pi(T) \).

(ii) Suppose now that \( \Gamma \), the character group of \( G \), is finitely generated with respect to the endomorphism \( A \), i.e. there exists a finite number of elements \( \gamma_1, \cdots, \gamma_r \) of \( \Gamma \) such that every element of \( \Gamma \) is of the form \( p_1(A)\gamma_1 + \cdots + p_r(A)\gamma_r \) where \( p_i \in \mathbb{Z}(A) \), the ring of polynomials in \( A \) with integer coefficients.

We can again suppose, without loss of generality, that \( T \) and \( A \) are one to one. This follows since \( \Gamma \) is a finitely generated module over the Noetherian ring \( \mathbb{Z}(A) \) and is therefore a Noetherian module [13]. Since \( \alpha(T) = \xi(F) \), \( \text{ann}(F) \) is a submodule of \( \Gamma \) and hence is finitely generated over \( \mathbb{Z}(A) \), i.e. there exists \( \beta_1, \cdots, \beta_s \in \Gamma \) such that

\[
\text{ann}(F) = \{ q_1(A)\beta_1 + \cdots + q_s(A)\beta_s \mid q_i \in \mathbb{Z}(A), i = 1, \cdots, s \}.
\]

Since \( A(\text{ann}(F)) = \text{ann}(F) \), \( \text{ann}(F) \), the character group of \( G/F \), is finitely generated with respect to \( A_{\alpha(T)} \). Since \( A_{\alpha(T)} \) and \( T_{\alpha(T)} \) are one to one and \( \pi(T) = \eta(T) \) if and only if \( \pi(T_{\alpha(T)}) = \eta(T_{\alpha(T)}) \), the assertion follows.

Let

\[
Y_m = \{ p_1(A)\gamma_1 + \cdots + p_r(A)\gamma_r \mid p_i \in \mathbb{Z}(A) \}
\]

and degree of \( p_i \leq m \), \( i = 1, \cdots, r \)

\( Y_m \) is a subgroup of \( \Gamma \) and \( \bigcup_{m=0}^{\infty} Y_m = \Gamma \). Choose \( m \) such that \( A^{-1}\gamma_i \in Y_m \), \( i = 1, \cdots, r \). Then \( A^{-1}Y_m \subset Y_m \) and if \( H = \text{ann}(Y_m) \), \( A^{-1}H \subset H \). Since \( A^n Y_m, n \geq 0 \), are finitely generated with no elements of finite order, \( G/A^{-n}H, n \geq 0 \), are finite-dimensional tori. We have \( \bigcup_{n=0}^{\infty} A^n Y_m = \Gamma \), \( \cap_{n=0}^{\infty} A^{-n}H = \{ 0 \} \) and \( \xi(H) \leq \xi(A^{-1}H) \leq \cdots \), \( V_{n=0}^{\infty} \xi(A^{-n}H) = \epsilon \). Since \( (T^{-1})_{\xi(A^{-n}H)} \) is an ergodic affine transformation of \( G/A^{-n}H \) we have \( \pi(T) = \pi(T^{-1}) = V_{n=0}^{\infty} \pi'((T^{-1})_{\xi(A^{-n}H)}) \) by Theorem 1, and if \( \Gamma_k' = \{ \gamma \in \Gamma \mid B^k \gamma = 0 \} \) where \( B = A - I \), then by part (i)

\[
\pi'((T^{-1})_{\xi(A^{-n}H)}) = \xi\left( \bigcup_{k=1}^{\infty} \Gamma_k' \cap A^n Y_m \right) \quad (\text{see } \S1.3).
\]
Therefore

\[ \pi(T) = \bigvee_{n=0}^{\infty} \tilde{\xi} \left( \text{ann} \left( \bigcup_{k=1}^{\infty} \Gamma_k' \cap A^n Y_n \right) \right) = \tilde{\xi} \left( \text{ann} \left( \bigcup_{k=1}^{\infty} \Gamma_k' \right) \right) = \eta(T). \]

(iii) Suppose now that \( G \) is a general compact connected metric abelian group. \( \Gamma \) is countable, \( \Gamma = \{ \gamma_1, \gamma_2, \ldots \} \). Let \( Y_n \) be the smallest subgroup of \( \Gamma \) containing \( \gamma_1, \ldots, \gamma_n \) and invariant relative to \( A \) (i.e. \( A Y_n \subseteq Y_n \)). If \( H_n = \text{ann}(Y_n) \), \( AH_n \subseteq H_n \) and \( T^{-1} \xi(H_n) \leq \xi(H_n) \).

We have

\[ Y_1 \subseteq Y_2 \subseteq Y_3 \subseteq \cdots, \quad \bigcup_{n=1}^{\infty} Y_n = \Gamma \]

\[ H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots, \quad \bigcap_{n=1}^{\infty} H_n = \{0\}, \]

and

\[ \xi(H_1) \leq \xi(H_2) \leq \xi(H_3) \leq \cdots, \quad \bigvee_{n=1}^{\infty} \xi(H_n) = \epsilon. \]

Using the fact that \( Y_n \) is finitely generated with respect to the endomorphism \( A_{\xi(H_n)} \) we have \( \pi'(T_{\xi(H_n)}) = \xi(\text{ann}(\bigcup_{k=1}^{\infty} \Gamma_k' \cap Y_n)) \) by part (ii) where \( \Gamma_k' = \{ \gamma \in \Gamma | B^k \gamma = 0 \} \), \( B = A - I \). (See §1.3.) Therefore by Theorem 1

\[ \pi(T) = \bigvee_{n=1}^{\infty} \pi'(T_{\xi(H_n)}) = \bigvee_{n=1}^{\infty} \xi \left( \text{ann} \left( \bigcup_{k=1}^{\infty} \Gamma_k' \cap Y_n \right) \right) = \xi \left( \text{ann} \left( \bigcup_{k=1}^{\infty} \Gamma_k' \right) \right) = \eta(T). \]

This completes the proof of the theorem.

Remark. To prove part (ii) we could have used a result of Laxton and Parry [5], which states that an automorphism \( A \) of a compact connected metric abelian group has a dense set of periodic points if the character group is finitely generated with respect to \( A \).

References


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