ON OKAMURA'S UNIQUENESS THEOREM

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Moyer [1] gives a uniqueness theorem for ordinary differential equations which includes many of the known criteria for uniqueness. It is demonstrated here that Moyer's results can be obtained as a special case of Okamura's theorem [2]. (See Yoshizawa, [3].)

If \( f(t, y) \) is defined and continuous on \( D: t_0 \leq t \leq t_0 + a, \ |y - y_0| \leq b, \) the problem of uniqueness for a solution \( z(t) \) to the initial value problem,

\[ y = f(t, y), \quad y(t_0) = y_0 \]

is equivalent to the uniqueness of the solution \( x(t) \equiv 0 \) of

\[ \dot{x} = F(t, x) = f(t, z(t)) - f(t, z(t) - x), \quad F(t, 0) = 0 \text{ on } D. \]

**Theorem 1** (Okamura [2], see also [3]). The \( x(t) \equiv 0 \) solution of (2) is unique to the right iff there exists a function \( V(t, x) \) defined on \( D \) such that (i) \( V(t, 0) = 0 \), (ii) \( V(t, x) > 0, \ x \neq 0 \), (iii) \( V(t, x) \) satisfies a local Lipschitz condition with respect to \( x \) and

\[
V'(t, x) = \liminf_{h \to 0} \frac{V(t + h, x + hF(t, x)) - V(t, x)}{h} \leq 0.
\]

Or, equivalently, the solution \( z(t) \) of (1) is unique if there exists a function \( V(t, x) \) such that \( V(t, z(t) - y) \) has the properties of Theorem 1. Moyer's results are obtained by defining

\[
V(t, z(t) - y) = \exp\left[ 2W(t, z(t) - y) \right].
\]

Moyer's Theorem 2.1 can be stated in the following manner.

**Theorem 2.** If there exists a \( V(t, x) \) as in Theorem 1 and \( z(t), \ y(t) \) are two distinct solutions of (1) then \( z(t) \equiv y(t) \) to the right of \( t_0 \) if and only if

\[
\liminf_{t \to t_0} V(t, z(t) - y(t)) = 0.
\]

**Remark.** Okamura's theorem shows it may be possible to have uniqueness of solutions of (1) but not assert the existence of the \( W(t, r) \) function of Moyer [1].

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References


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