COMPARISON AND APPLICATION OF TWO GREEN’S MATRICES

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1. Purpose. The purpose of this paper is to demonstrate a close but not obvious similarity between two Green’s matrices due respectively to W. M. Whyburn and R. H. Cole and then to apply these matrices to the solution of a difference system.

2. Whyburn’s Green’s matrix. Whyburn [1] has given a Green’s matrix,

\[ G_w(x, t) = \begin{cases} 
U(x)H_1(U) \left[ AU(a) + \int_a^t F(s)U(s)ds \right] U^I(t), & t < x, \\
- U(x)H_1(U) \left[ BU(b) + \int_t^b F(s)U(s)ds \right] U^I(t), & t > x, 
\end{cases} \]

for the system,

\[
\begin{align*}
(1) & \quad L_1(Y) = Y'(x) + P(x)Y(x) = 0, \\
(2) & \quad H_1(Y) = AY(a) + BY(b) + \int_a^b F(x)Y(x)dx = 0.
\end{align*}
\]

This Green’s matrix yields a solution of the nonhomogeneous \( L_1(Y) = Q(x) \), \( H_1(Y) = D \), assuming \( A, B, D \) constant \( n \times n \) matrices and \( P, Q, F \) matrices of Lebesgue summable functions. \( U \) is nonsingular on \([a, b]\) and \( U'(x) + P(x)U(x) = 0 \). The superscript \( I \) is used to indicate matrix inverse as opposed to operator inverse.

Whyburn [2] has shown that if we replace the boundary condition \( H_1(Y) = D \) with the more general

\[
\sum_{d_i \in q} A_iY(d_i) + \int_a^b F_1(x)Y(x)dx = D_1,
\]

where \( q \) is a first species subset of \([a, b]\), then there are matrices \( A, B, D, \) and \( F(x) \) such that the nonhomogeneous system associated with (1), (2) is equivalent to this new system.


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\[ G_c(x, t) = \int_a^t U(x)H^I_2(U)dF(s)U(s)U^I(t), \quad t < x, \]
\[ = - \int_a^b U(x)H^I_2(U)dF(s)U(s)U^I(t), \quad t > x, \]
yields a solution of
\begin{align*}
(4) & \quad L_2(Y) = Y'(x) = A(x)Y(x) + B(x), \\
(5) & \quad H_2(Y) = \sum_{i=1}^{m} W_i Y(a_i) + \int_a^b W(x)Y(x)dx = D,
\end{align*}
while being completely determined by the homogeneous \( L_2(Y) = 0, H_2(Y) = 0 \), a different system from (1), (2). \( a = a_1 < a_2 < \cdots < a_m = b \), and \( F(s) \) is the sum of \( F_2(s) = \int_a^b W(x)dx \) and the step-function \( F_1(s) \), with \( F_1(a) = 0 \) and \( F_1(a^+) - F_1(a^-) = W_h \).

The reader will not find it unduly difficult to show that
\[ G_c(x, t) = U(x)H^I_2(U) \left[ \sum_{i=1}^{p} W_i U(a_i) + \int_a^t W(s)U(s)ds \right]U^I(t), \quad t < x, \]
\[ = - U(x)H^I_2(U) \left[ \sum_{i=1}^{m} W_i U(a_i) + \int_t^b W(s)U(s)ds \right]U^I(t), \quad t > x, \]
where \( a_p \leq t \leq a_{p+1} \).

4. Careful examination of Whyburn's results [2] reveals that for each system (4), (5) there are several equivalent systems of the type \( L_1(Y) = Q(x), H_1(Y) = D \). One of particular interest is found by defining \( R(x) = \sum_i W_i A(x) \) on \( a_j < x \leq a_{j+1} \), so that
\begin{align*}
\int_a^b R(x)Y(x)dx &= \sum_{j=1}^{m-1} \sum_{i=1}^{j} W_i \int_{a_i}^{a_{i+1}} \left[ Y'(x) - B(x) \right]dx \\
&= \sum_{j=1}^{m-1} \sum_{i=1}^{j} W_i \left[ Y(a_{j+1}) - Y(a_j) \right] \\
&\quad - \sum_{j=1}^{m-1} \sum_{i=1}^{j} W_i \int_{a_j}^{a_{j+1}} B(x)dx \\
&= - \sum_{i=1}^{m} W_i Y(a_i) + \sum_{i=1}^{m} W_i Y(b) \\
&\quad - \sum_{j=1}^{m-1} \sum_{i=1}^{j} W_i \int_{a_j}^{a_{j+1}} B(x)dx.
\end{align*}
System (4), (5) is now seen to be equivalent to the system which results from replacing the boundary condition \( H_1(Y) = D \) by

\[
H_3(Y) = \sum_{i=1}^{m} W_i Y(b) + \int_{a}^{b} \left[ W(x) - R(x) \right] Y(x) dx = D
\]

(6)

\[
+ \sum_{j=1}^{m-1} \sum_{i=1}^{j} W_i \int_{a_j}^{a_{j+1}} B(x) dx,
\]

which is a special case of Whyburn’s endpoint-integral condition.

It can be shown that Whyburn’s Green’s matrix for \( L_1(Y) = 0, H_3(Y) = 0 \) is given by

\[
G_W(x, t) = U(x) H_2^I(U) \left[ \int_{a}^{t} W(x) U(x) dx + \sum_{i=1}^{p} W_i U(a_i) \right.
\]

\[
- \sum_{1}^{p} W_i U(l) \right] U^I(t), \quad t < x,
\]

\[
= - U(x) H_2^I(U) \left[ \int_{t}^{b} W(x) U(x) dx + \sum_{p+1}^{m} W_i U(a_i) \right.
\]

\[
+ \sum_{1}^{p} W_i U(l) \right] U^I(t), \quad t > x,
\]

which, since (1), (6) and (4), (5) are equivalent, also provides the solution of (4), (5).

It is noteworthy that \( G_W \) and \( G_C \) share the discontinuity

\[
G(x, x^-) - G(x, x^+) = E
\]

along the line \( x = t \), and, while \( G_W \) is otherwise continuous on \([a, b] \times [a, b]\), \( G_C \) has discontinuities along the lines \( t = a_1, \ldots, a_m \), given by

\[
G_C(x, a_i^+) - G_C(x, a_i^-) = U(x) H_2^I(U) W_{a_i}, \quad \text{where we interpret } G_C(x, a_i^-) \text{ and } G_C(x, a_i^+) \text{ as } 0.
\]

The terms \( \pm \sum_i W_i U(l) \), which in \( G_W \) absorb those latter mentioned discontinuities of \( G_C \), are readily seen to be the only actual difference between \( G_W \) and \( G_C \).

5. **The difference system.** Several authors, including Whyburn [4], have given Green’s matrices which yield the solution of the difference system

\[
(Y_{i+1} - Y_i)/(x_{i+1} - x_i) = R_i Y_i + S_i, \quad i = 1, 2, \ldots, m - 1,
\]

(7)

\[
A Y_0 + B Y_m = C.
\]

(8)

A boundary condition more nearly analogous to \( H_1(Y) = D \) or (5) is
The system (7), (9) is more complex, and we now offer a theorem which will bring the powerful theory of Whyburn (Cole) to bear on this system.

Suppose that for \(0 \leq x_i \leq x_{i+1} - x_i\), the matrix \(z_iR_i + E\) is nonsingular. Let \(U_0 = E\) and for \(i > 0\),

\[
U_i = \prod_{j=i-1}^{0} [(x_{j+1} - x_j)R_j + E],
\]

and assume \(\sum_0^m A_j U_j\) is nonsingular (necessary and sufficient for the uniqueness of solution of (7), (9)).

**Theorem.** If \(P^*(x) = -R_i[(x - x_i)R_i + E]^T\) and \(Q^*(x) = -(x - x_i)S_i + S_i\) for \(x_i \leq x \leq x_{i+1}\), then the Whyburn's Green's matrix for

\[
Y^*(x)' + P^*(x)Y^*(x) = 0,
\]
\[
H^*(Y^*) = \sum_0^m A_j Y^*(x_j) = 0
\]

yields the unique solution \(Y^*\) of

\[
Y^*(x)' + P^*(x)Y^*(x) = Q^*(x),
\]
\[
H^*(Y^*) = \sum_0^m A_j Y^*(x) = C
\]

and \(Y_i = Y^*(x_i)\) is the unique solution of (7), (9).

**Remark.** The requirement that \(z_iR_i + E\) be nonsingular is not so restrictive as it may appear. If, for example, (7), (9) is an approximation to a system such as (1), (2), with (after Whyburn [4])

\[
R_i = \int_{x_i}^{x_{i+1}} P(x)dx/(x_{i+1} - x_i),
\]

then the required nonsingularity is automatic for sufficiently fine subdivision of \([a, b]\).

**Proof of Theorem.** Let

\[
Y^*(x) = Y_i + [Y_{i+1} - Y_i](x - x_i)/(x_{i+1} - x_i)
\]

\[
= [(x - x_i)R_i + E]Y_i + (x - x_i)S_i
\]
so that \( Y^*(x) = R_i Y_i + S_i \) on \( x_i < x < x_{i+1} \). Solving for \( Y_i \), we get

\[
Y_i = [(x - x_i)R_i + E]^I Y^*(x) - [(x - x_i)R_i + E]^I (x - x_i)S_i
\]

so that

\[
Y^*(x) = R_i [(x - x_i)R_i + E]^I Y^*(x)
- R_i [(x - x_i)R_i + E]^I (x - x_i)S_i + S_i,
\]

which is to say that \( Y^* + P^* Y^* = Q^* \).

If

\[
U^*(x) = [(x - x_i)R_i + E] \prod_{j=i-1}^0 [R_j (x_{j+1} - x_j) + E]
\]
on \( x_i < x < x_{i+1} \), then \( U^*(x) + P^*(x) U^*(x) = 0 \) and \( U^* \) is nonsingular an \([a, b]\), and \( U^*(x_i) = U_i \) so that \( \det [H^*(U^*)] = \det [\sum_0^m A_j U_j] \neq 0 \) which implies the incompatibility of the homogeneous system, and is sufficient for the existence of Whyburn’s Green’s matrix for (10), (11).

One easily verifies that \( Y_i = \bar{Y}^*(x_i) \) is the required solution of (7), (9), and this completes the proof.

Remark. One can show that if \( x_s < t < x_{s+1} \) and \( x_i < x < x_{i+1} \), then

\[
G_w(x, t) = [(x - x_i)R_i + E] G_{is} [(x_{s+1} - x_i)R_s + E] [(t - x_i)R_s + E]^I
- [(x - x_i)R_i + E] U_i H_3^I(U) \sum_{j=0}^s A_j,
\]

(where \( G_{is} \) is a difference Green’s matrix yielding solution of (7), (9)) holds for all \((x, t)\) except on the interiors of triangles with vertices of the form \( \{(x_i, x_i), (x_{s+1}, x_{s+1}), (x_{s+1}, x_i)\} \).

The author regrets that he is able to establish the above assertion only after a laborious computation, which does not seem appropriate for inclusion here. The above-mentioned Green’s matrix is

\[
G_{is} = \sum_{j=0}^s U_i H_3^I(U) A_j U_j U_{s+1}^I, \quad i > s,
\]

\[
- \sum_{j=s+1}^m U_i H_3^I(U) A_j U_j U_{s+1}^I, \quad i \leq s,
\]

and it is this writer’s intention to make known certain interesting properties of this matrix at a later date.

Bibliography

We shall study the asymptotic behavior for $t \to \infty$ of solutions of the following nonlinear differential equation:

\[
(1) \quad u'' + f(t, u) = 0.
\]

We suppose that $f(t, u)$ satisfies the following conditions:

H-1: $f(t, u)$ is continuous in $D: t \geq 0, -\infty < u < \infty$.

H-2: The derivative $f_u$ exists on $D$ and satisfies $f_u(t, u) > 0$ on $D$.

H-3: $|f(t, u(t))| \leq f_u(t, 0)|u(t)|$ on $D$.

An important class of functions $f(t, u)$ which satisfy conditions H-1, 2, 3 is the class of twice continuously differentiable functions $f(t, u)$ which are odd and strictly monotone in $u$ with $f_{uu} \geq 0$ for $u < 0$ and $f_{uu} \leq 0$ for $u > 0$. Nonlinear eigenvalue problems involving this class of functions have been studied extensively by G. H. Pimbley [1].

For the case $f(t, u) = \pm t^n u^n$, R. Bellman [2] has given an exhaustive treatment of the asymptotic behavior of proper solutions (i.e., solutions which exist and have continuous derivatives for $t \geq t_0$). For the case $f(t, u) = a(t)u^{2n+1}$ several results on asymptotic behavior exist depending on properties of $a(t)$. References can be found in the papers of P. Waltman [3] and R. A. Moore and Z. Nehari [4].

Our basic result is that there exist solutions of (1) which approach those of $u'' = 0$. More precisely, we prove the