COMPARISON AND APPLICATION OF TWO GREEN'S MATRICES

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1. Purpose. The purpose of this paper is to demonstrate a close but not obvious similarity between two Green's matrices due respectively to W. M. Whyburn and R. H. Cole and then to apply these matrices to the solution of a difference system.

2. Whyburn's Green's matrix. Whyburn [1] has given a Green's matrix,

\[ G_w(x, t) = U(x) H^I(U) \left[ A U(a) + \int_a^t F(s) U(s) ds \right] U^I(t), \quad t < x, \]

\[ = - U(x) H^I(U) \left[ B U(b) + \int_t^b F(s) U(s) ds \right] U^I(t), \quad t > x, \]

for the system,

\begin{align*}
L_1(Y) &= Y'(x) + P(x) Y(x) = 0, \\
H_1(Y) &= A Y(a) + B V(b) + \int_a^b F(x) Y(x) dx = 0.
\end{align*}

This Green's matrix yields a solution of the nonhomogeneous \( L_1(Y) = Q(x), \ H_1(Y) = D, \) assuming \( A, B, D \) constant \( n \times n \) matrices and \( P, Q, F \) matrices of Lebesgue summable functions. \( U \) is nonsingular on \( [a, b] \) and \( U'(x) + P(x) U(x) = 0 \). The superscript \( I \) is used to indicate matrix inverse as opposed to operator inverse.

Whyburn [2] has shown that if we replace the boundary condition \( H_1(Y) = D \) with the more general

\[ \sum_{d_i \in q} A_{d_i} Y(d_i) + \int_a^b F_1(x) Y(x) dx = D_1, \]

where \( q \) is a first species subset of \( [a, b] \), then there are matrices \( A, B, D, \) and \( F(x) \) such that the nonhomogeneous system associated with (1), (2) is equivalent to this new system.


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\[ G_c(x, t) = \int_a^t U(x)H_2^I(U)dF(s)U(s)U^I(t), \quad t < x, \]
\[ = - \int_t^b U(x)H_2^I(U)dF(s)U(s)U^I(t), \quad t > x, \]
yields a solution of
\[ L_2(Y) = Y'(x) = A(x)Y(x) + B(x), \quad (4) \]
\[ H_2(Y) = \sum_{i=1}^m W_i Y(a_i) + \int_a^b W(x)Y(x)dx = D, \quad (5) \]
while being completely determined by the homogeneous \( L_2(Y) = 0, \)
\( H_2(Y) = 0, \) a different system from \((1), (2). \) \( a = a_1 < a_2 < \cdots < a_m = b, \)
and \( F(s) \) is the sum of \( F_2(s) = \int_a^s W(x)dx \) and the step-function \( F_1(s), \)
with \( F_1(a) = 0 \) and \( F_1(a^+_{\infty}) - F_1(a^-_{\infty}) = W_h. \)

The reader will not find it unduly difficult to show that
\[ G_c(x, t) = \left[ \sum_{i=1}^p W_i U(a_i) + \int_a^t W(s)U(s)ds \right]U^I(t), \quad t < x, \]
\[ = - \left[ \sum_{i=p+1}^m W_i U(a_i) + \int_t^b W(s)U(s)ds \right]U^I(t), \quad t > x, \]
where \( a_p \leq t \leq a_{p+1}. \)

each system \((4), (5)\) there are several equivalent systems of the type
\( L_1(Y) = Q(x), H_1(Y) = D. \) One of particular interest is found by defining \( R(x) = \sum_i W_i A(x) \) on \( a_j < x \leq a_{j+1}, \) so that
\[ \int_a^b R(x) Y(x)dx = \sum_{j=1}^{m-1} \sum_{i=1}^j W_i \int_{a_j}^{a_{j+1}} [Y'(x) - B(x)]dx \]
\[ = \sum_{j=1}^{m-1} \sum_{i=1}^j W_i [Y(a_{j+1}) - Y(a_j)] \]
\[ - \sum_{j=1}^{m-1} \sum_{i=1}^j W_i \int_{a_j}^{a_{j+1}} B(x)dx \]
\[ = - \sum_{i=1}^m W_i Y(a_i) + \sum_{i=1}^m W_i Y(b) \]
\[ - \sum_{j=1}^{m-1} \sum_{i=1}^j W_i \int_{a_j}^{a_{j+1}} B(x)dx. \]
System (4), (5) is now seen to be equivalent to the system which results from replacing the boundary condition $H_1(Y) = D$ by

$$H_3(Y) = \sum_{i=1}^{m} W_i Y(b) + \int_{a}^{b} \left[ W(x) - R(x) \right] Y(x) dx = D$$

(6)

$$+ \sum_{j=1}^{m-1} \sum_{i=1}^{j} W_i \int_{a_j}^{a_{j+1}} B(x) dx,$$

which is a special case of Whyburn’s endpoint-integral condition.

It can be shown that Whyburn’s Green’s matrix for $L_1(Y) = 0$, $H_3(Y) = 0$ is given by

$$G_W(x, t) = U(x) H_2^f(U) \left[ \int_{a}^{t} W(x) U(x) dx + \sum_{i=1}^{p} W_i U(a_i) \right. \left. - \sum_{i=1}^{p} W_i U(l_i) \right] U^T(t), \quad t < x,$$

$$= - U(x) H_2^f(U) \left[ \int_{t}^{b} W(x) U(x) dx + \sum_{p+1}^{m} W_i U(a_i) \right. \left. + \sum_{i=1}^{p} W_i U(l_i) \right] U^T(t), \quad t > x,$$

which, since (1), (6) and (4), (5) are equivalent, also provides the solution of (4), (5).

It is noteworthy that $G_W$ and $G_C$ share the discontinuity

$$G(x, x^-) - G(x, x^+) = E$$

along the line $x = t$, and, while $G_W$ is otherwise continuous on $[a, b] \times [a, b]$, $G_C$ has discontinuities along the lines $t = a_1, \ldots, a_m$, given by $G_C(x, a_i^+) - G_C(x, a_i^-) = U(x) H_2^f(U) W_i$, where we interpret $G_C(x, a_i^-)$ and $G_C(x, a_i^+)$ as 0. The terms $\pm \sum_{i=1}^{p} W_i U(l_i)$, which in $G_W$ absorb those latter mentioned discontinuities of $G_C$, are readily seen to be the only actual difference between $G_W$ and $G_C$.

5. The difference system. Several authors, including Whyburn [4], have given Green’s matrices which yield the solution of the difference system

(7) \( (Y_{i+1} - Y_i)/(x_{i+1} - x_i) = R_i Y_i + S_i, \quad i = 1, 2, \ldots, m - 1, \)

(8) \( A Y_0 + B Y_m = C. \)

A boundary condition more nearly analogous to $H_1(Y) = D$ or (5) is
The system (7), (9) is more complex, and we now offer a theorem which will bring the powerful theory of Whyburn (Cole) to bear on this system.

Suppose that for \( 0 \leq z_i \leq x_{i+1} - x_i \), the matrix \( z_i R_i + E \) is nonsingular. Let \( U_0 = E \) and for \( i > 0 \),

\[
U_i = \prod_{j=i-1}^{0} [(x_{j+1} - x_j) R_j + E],
\]

and assume \( \sum_{0}^{m} A_j U_j \) is nonsingular (necessary and sufficient for the uniqueness of solution of (7), (9)).

**Theorem.** If \( P^*(x) = -R_i [(x - x_i) R_i + E] \) and \( Q^*(x) = -R_i [(x - x_i) R_i + E]^T (x - x_i) S_i + S_i \)

for \( x_i \leq x \leq x_{i+1} \), then the Whyburn’s Green’s matrix for

\[
Y^*(x') + P^*(x) Y^*(x) = 0,
\]

\[
H^*(Y^*) = \sum_{0}^{m} A_j Y^*(x_j) = 0
\]

yields the unique solution \( Y^* \) of

\[
Y^*(x') + P^*(x) Y^*(x) = Q^*(x),
\]

\[
H^*(Y^*) = \sum_{0}^{m} A_j Y^*(x_j) = C
\]

and \( Y_i = Y^*(x_i) \) is the unique solution of (7), (9).

**Remark.** The requirement that \( z_i R_i + E \) be nonsingular is not so restrictive as it may appear. If, for example, (7), (9) is an approximation to a system such as (1), (2), with (after Whyburn [4])

\[
R_i = \int_{x_i}^{x_{i+1}} P(x) dx/(x_{i+1} - x_i),
\]

then the required nonsingularity is automatic for sufficiently fine subdivision of \([a, b]\).

**Proof of Theorem.** Let

\[
Y^*(x) = Y_i + [Y_{i+1} - Y_i] (x - x_i)/(x_{i+1} - x_i)
\]

\[
= [(x - x_i) R_i + E] Y_i + (x - x_i) S_i
\]
so that \( Y^*(x) = R_i Y_i + S_i \) on \( x_i \leq x < x_{i+1} \). Solving for \( Y_i \), we get

\[
Y_i = [(x - x_i)R_i + E]^I Y(x) - [(x - x_i)R_i + E]^I (x - x_i)S_i
\]

so that

\[
Y^*(x) = R_i[(x - x_i)R_i + E]^I Y(x)
- R_i[(x - x_i)R_i + E]^I (x - x_i)S_i + S_i,
\]

which is to say that \( Y^* + P^* Y^* = Q^* \).

If

\[
U^*(x) = [(x - x_i)R_i + E] \prod_{j=i-1}^{0} [R_j(x_{j+1} - x_j) + E]
\]

on \( x_i \leq x < x_{i+1} \), then \( U^*(x) + P^*(x) U^*(x) = 0 \) and \( U^* \) is nonsingular an \([a, b]\), and \( U^*(x_i) = U_i \) so that \( \det [H^*(U^*)] = \det [\sum_{i=0}^{m} A_j U_j] \neq 0 \) which implies the incompatibility of the homogeneous system, and is sufficient for the existence of Whyburn’s Green’s matrix for (10), (11).

One easily verifies that \( Y_i = Y^*(x_i) \) is the required solution of (7), (9), and this completes the proof.

**Remark.** One can show that if \( x_s \leq t < x_{s+1} \) and \( x_i \leq x < x_{i+1} \), then

\[
G_{w}(x, t) = [(x - x_i)R_i + E] G_{i,s} [(x_{s+1} - x_i)R_s + E] [(t - x_s)R_s + E]^T
- [(x - x_i)R_i + E] U_i H_3(U) \sum_{j=0}^{0} A_j,
\]

(where \( G_{i,s} \) is a difference Green’s matrix yielding solution of (7), (9)) holds for all \((x, t)\) except on the interiors of triangles with vertices of the form \( \{(x_i, x_i), (x_{i+1}, x_{i+1}), (x_{i+1}, x_i)\} \).

The author regrets that he is able to establish the above assertion only after a laborious computation, which does not seem appropriate for inclusion here. The above-mentioned Green’s matrix is

\[
G_{is} = \sum_{j=0}^{s} U_i H_3(U) A_j U_j U_{s+1}^T, \quad i > s,
\]

\[
- \sum_{j=s+1}^{m} U_i H_3(U) A_j U_j U_{s+1}^T, \quad i \leq s,
\]

and it is this writer’s intention to make known certain interesting properties of this matrix at a later date.

**Bibliography**


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THE ASYMPTOTIC BEHAVIOR OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS

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We shall study the asymptotic behavior for \( t \to \infty \) of solutions of the following nonlinear differential equation:

\[
 u'' + f(t, u) = 0.
\]

We suppose that \( f(t, u) \) satisfies the following conditions:

H-1: \( f(t, u) \) is continuous in \( D: t \geq 0, -\infty < u < \infty \).

H-2: The derivative \( f_u \) exists on \( D \) and satisfies \( f_u(t, u) > 0 \) on \( D \).

H-3: \( |f(t, u(t))| \leq f_u(t, 0)|u(t)| \) on \( D \).

An important class of functions \( f(t, u) \) which satisfy conditions H-1, 2, 3 is the class of twice continuously differentiable functions \( f(t, u) \) which are odd and strictly monotone in \( u \) with \( f_{uu} \geq 0 \) for \( u < 0 \) and \( f_{uu} \leq 0 \) for \( u > 0 \). Nonlinear eigenvalue problems involving this class of functions have been studied extensively by G. H. Pimbley [1].

For the case \( f(t, u) = \pm t^n u^n \), R. Bellman [2] has given an exhaustive treatment of the asymptotic behavior of proper solutions (i.e., solutions which exist and have continuous derivatives for \( t \geq t_0 \)). For the case \( f(t, u) = a(t)u^{2n+1} \) several results on asymptotic behavior exist depending on properties of \( a(t) \). References can be found in the papers of P. Waltman [3] and R. A. Moore and Z. Nehari [4].

Our basic result is that there exist solutions of (1) which approach those of \( u'' = 0 \). More precisely, we prove the