COMPARISON AND APPLICATION OF TWO GREEN'S MATRICES

A. L. DEAL, III

1. **Purpose.** The purpose of this paper is to demonstrate a close but not obvious similarity between two Green's matrices due respectively to W. M. Whyburn and R. H. Cole and then to apply these matrices to the solution of a difference system.

2. **Whyburn's Green's matrix.** Whyburn [1] has given a Green's matrix,

$$G_W(x,t) = U(x)H_1^I(U)\left[AU(a) + \int_a^t F(s)U(s)ds\right]U^I(t), \qquad t < x,$$

$$= -\,U(x)H_1^I(U)\left[BU(b) + \int_t^b F(s)U(s)ds\right]U^I(t), \qquad t > x,$$

for the system,

$$L_1(Y) = Y'(x) + P(x)Y(x) = 0, \tag{1}$$

$$H_1(Y) = AY(a) + BY(b) + \int_a^b F(x)Y(x)dx = 0. \tag{2}$$

This Green's matrix yields a solution of the nonhomogeneous $L_1(Y) = Q(x)$, $H_1(Y) = D$, assuming A, B, D constant $n \times n$ matrices and P, Q, F matrices of Lebesgue summable functions. U is nonsingular on $[a, b]$ and $U'(x) + P(x)U(x) = 0$. The superscript I is used to indicate matrix inverse as opposed to operator inverse.

Whyburn [2] has shown that if we replace the boundary condition $H_1(Y) = D$ with the more general

$$\sum_{d_i \in q} A_i Y(d_i) + \int_a^b F_1(x)Y(x)dx = D_1, \tag{3}$$

where q is a first species subset of $[a, b]$, then there are matrices A, B, D, and $F(x)$ such that the nonhomogeneous system associated with (1), (2) is equivalent to this new system.

3. **Cole's Green's matrix.** Cole [3] has discovered that the Green's matrix

Received by the editors July 15, 1966.

$$G_C(x, t) = \int_a^t U(x)H_2^I(U)dF(s)U(s)U^I(t), \qquad t < x,$$
$$= -\int_t^b U(x)H_2^I(U)dF(s)U(s)U^I(t), \quad t > x,$$

yields a solution of

$$L_2(Y) = Y'(x) = A(x)Y(x) + B(x), \tag{4}$$

$$H_2(Y) = \sum_1^m W_h Y(a_h) + \int_a^b W(x)Y(x)dx = D, \tag{5}$$

while being completely determined by the homogeneous $L_2(Y)=0$, $H_2(Y)=0$, a different system from (1), (2). $a=a_1<a_2< \cdots <a_m=b$, and $F(s)$ is the sum of $F_2(s)=\int_a^s W(x)dx$ and the step-function $F_1(s)$, with $F_1(a)=0$ and $F_1(a_h^+)-F_1(a_h^-)=W_h$.

The reader will not find it unduly difficult to show that

$$G_C(x, t) = U(x)H_2^I(U)\left[\sum_1^p W_iU(a_i) + \int_a^t W(s)U(s)ds\right]U^I(t), \quad t < x,$$
$$= -U(x)H_2^I(U)\left[\sum_{p+1}^m W_iU(a_i) + \int_t^b W(s)U(s)ds\right]U^I(t),$$
$$t > x,$$

where $a_p \leqq t \leqq a_{p+1}$.

-4. Careful examination of Whyburn's results [2] reveals that for each system (4), (5) there are several equivalent systems of the type $L_1(Y)=Q(x)$, $H_1(Y)=D$. One of particular interest is found by defining $R(x)=\sum_1^j W_iA(x)$ on $a_j<x\leqq a_{j+1}$, so that

$$\int_a^b R(x)Y(x)dx = \sum_{j=1}^{m-1}\sum_{i=1}^{j} W_i \int_{a_j}^{a_{j+1}} [Y'(x) - B(x)]dx$$
$$= \sum_{j=1}^{m-1}\sum_{i=1}^{j} W_i[Y(a_{j+1}) - Y(a_j)]$$
$$- \sum_{j=1}^{m-1}\sum_{i=1}^{j} W_i \int_{a_j}^{a_{j+1}} B(x)dx$$
$$= -\sum_1^m W_iY(a_i) + \sum_1^m W_iY(b)$$
$$- \sum_{j=1}^{m-1}\sum_{i=1}^{j} W_i \int_{a_j}^{a_{j+1}} B(x)dx.$$

System (4), (5) is now seen to be equivalent to the system which results from replacing the boundary condition $H_1(Y)=D$ by

$$
\begin{aligned}
H_3(Y) &= \sum_1^m W_i Y(b) + \int_a^b [W(x) - R(x)]Y(x)dx = D \\
&\quad + \sum_{j=1}^{m-1}\sum_{i=1}^{j} W_i \int_{a_j}^{a_{j+1}} B(x)dx,
\end{aligned}
\tag{6}
$$

which is a special case of Whyburn's endpoint-integral condition.

It can be shown that Whyburn's Green's matrix for $L_1(Y)=0$, $H_3(Y)=0$ is given by

$$
\begin{aligned}
G_W(x, t) &= U(x)H_2^I(U)\left[\int_a^t W(x)U(x)dx + \sum_1^p W_i U(a_i) \right. \\
&\qquad \left. - \sum_1^p W_i U(t)\right] U^I(t), \qquad t < x, \\
&= -\,U(x)H_2^I(U)\left[\int_t^b W(x)U(x)dx + \sum_{p+1}^m W_i U(a_i) \right. \\
&\qquad \left. + \sum_1^p W_i U(t)\right] U^I(t), \qquad t > x,
\end{aligned}
$$

which, since (1), (6) and (4), (5) are equivalent, also provides the solution of (4), (5).

It is noteworthy that G_W and G_C share the discontinuity

$$G(x, x^-) - G(x, x^+) = E$$

along the line $x=t$, and, while G_W is otherwise continuous on $[a, b] \times [a, b]$, G_C has discontinuities along the lines $t=a_1, \cdots, a_m$, given by $G_C(x, a_h^+) - G_C(x, a_h^-) = U(x)H_2^I(U)W_h$, where we interpret $G_C(x, a_1^-)$ and $G_C(x, a_m^+)$ as 0. The terms $\pm \sum_1^p W_i U(t)$, which in G_W absorb those latter mentioned discontinuities of G_C, are readily seen to be the only actual difference between G_W and G_C.

5. The difference system. Several authors, including Whyburn [4], have given Green's matrices which yield the solution of the difference system

$$(Y_{i+1} - Y_i)/(x_{i+1} - x_i) = R_i Y_i + S_i, \qquad i = 1, 2, \cdots, m-1, \tag{7}$$

$$A Y_0 + B Y_m = C. \tag{8}$$

A boundary condition more nearly analogous to $H_1(Y)=D$ or (5) is

(9) $$H_3(Y) = \sum_0^m A_jY_j = C.$$

The system (7), (9) is more complex, and we now offer a theorem which will bring the powerful theory of Whyburn (Cole) to bear on this system.

Suppose that for $0 \leqq z_i \leqq x_{i+1} - x_i$, the matrix $z_iR_i + E$ is nonsingular. Let $U_0 = E$ and for $i > 0$,

$$U_i = \prod_{j=i-1}^{0} [(x_{j+1} - x_j)R_j + E],$$

and assume $\sum_0^m A_jU_j$ is nonsingular (necessary and sufficient for the uniqueness of solution of (7), (9)).

THEOREM. *If* $P^*(x) = -R_i[(x - x_i)R_i + E]^I$ *and*

$$Q^*(x) = -R_i[(x - x_i)R_i + E]^I(x - x_i)S_i + S_i$$

for $x_i \leqq x \leqq x_{i+1}$, *then the Whyburn's Green's matrix for*

(10) $$Y^{*\prime}(x) + P^*(x)Y^*(x) = 0,$$

(11) $$H^*(Y^*) = \sum_0^m A_jY^*(x_j) = 0$$

yields the unique solution Y^* *of*

$$Y^{*\prime}(x) + P^*(x)Y^*(x) = Q^*(x),$$

$$H^*(Y^*) = \sum_0^m A_jY^*(x) = C$$

and $Y_i = Y^*(x_i)$ *is the unique solution of* (7), (9).

REMARK. The requirement that $z_iR_i + E$ be nonsingular is not so restrictive as it may appear. If, for example, (7), (9) is an approximation to a system such as (1), (2), with (after Whyburn [4])

$$R_i = \int_{x_i}^{x_{i+1}} P(x)dx/(x_{i+1} - x_i),$$

then the required nonsingularity is automatic for sufficiently fine subdivision of $[a, b]$.

PROOF OF THEOREM. Let

$$\begin{aligned} Y^*(x) &= Y_i + [Y_{i+1} - Y_i](x - x_i)/(x_{i+1} - x_i) \\ &= [(x - x_i)R_i + E]Y_i + (x - x_i)S_i \end{aligned}$$

so that $Y^{*\prime}(x)=R_iY_i+S_i$ on $x_i\leqq x<x_{i+1}$. Solving for Y_i, we get

$$Y_i = [(x-x_i)R_i+E]^I Y^*(x) - [(x-x_i)R_i+E]^I(x-x_i)S_i$$

so that

$$\begin{aligned} Y^{*\prime}(x) &= R_i[(x-x_i)R_i+E]^I Y^*(x) \\ &\quad - R_i[(x-x_i)R_i+E]^I(x-x_i)S_i + S_i, \end{aligned}$$

which is to say that $Y^{*\prime}+P^*Y^*=Q^*$.

If

$$U^*(x) = [(x-x_i)R_i+E]\prod_{j=i-1}^{0}[R_j(x_{j+1}-x_j)+E]$$

on $x_i\leqq x<x_{i+1}$, then $U^{*\prime}(x)+P^*(x)U^*(x)=0$ and U^* is nonsingular an $[a, b]$, and $U^*(x_i)=U_i$ so that $\det[H^*(U^*)]=\det[\sum_0^m A_jU_j]\neq 0$ which implies the incompatibility of the homogeneous system, and is sufficient for the existence of Whyburn's Green's matrix for (10), (11).

One easily verifies that $Y_i=Y^*(x_i)$ is the required solution of (7), (9), and this completes the proof.

REMARK. One can show that if $x_s\leqq t<x_{s+1}$ and $x_i\leqq x<x_{i+1}$, then

$$\begin{aligned} G_W(x,t) &= [(x-x_i)R_i+E]G_{is}[(x_{s+1}-x_s)R_s+E][(t-x_s)R_s+E]^I \\ &\quad - [(x-x_i)R_i+E]U_i\overset{I}{H_3}(U)\sum_0^s A_j, \end{aligned}$$

(where G_{is} is a difference Green's matrix yielding solution of (7), (9)) holds for all (x, t) except on the interiors of triangles with vertices of the form $\{(x_i, x_i), (x_{i+1}, x_{i+1}), (x_{i+1}, x_i)\}$.

The author regrets that he is able to establish the above assertion only after a laborious computation, which does not seem appropriate for inclusion here. The above-mentioned Green's matrix is

$$\begin{aligned} G_{is} &= \sum_{j=0}^{s} U_i\overset{I}{H_3}(U)A_jU_j\overset{I}{U}_{s+1}, \qquad && i>s, \\ &\quad - \sum_{j=s+1}^{m} U_i\overset{I}{H_3}(U)A_jU_j\overset{I}{U}_{s+1}, && i\leqq s, \end{aligned}$$

and it is this writer's intention to make known certain interesting properties of this matrix at a later date.

BIBLIOGRAPHY

1. W. M. Whyburn, *Differential equations with general boundary conditions*, Bull. Amer. Math. Soc. **48** (1942), 692–704.

2. ———, *Differential systems with general boundary conditions*, Seminar Reports in Mathematics, University of California Publications in Mathematics **2** (1944), 45–61.

3. R. H. Cole, *General boundary conditions for an ordinary linear differential system*, Trans. Amer. Math. Soc. **111** (1964), 521–550.

4. W. M. Whyburn, *On related difference and differential systems*, Amer. J. Math. **51** (1929), 265–286.

THE VIRGINIA MILITARY INSTITUTE

THE ASYMPTOTIC BEHAVIOR OF A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS[1]

DONALD S. COHEN

We shall study the asymptotic behavior for $t\to\infty$ of solutions of the following nonlinear differential equation:

$$u'' + f(t, u) = 0. \tag{1}$$

We suppose that $f(t, u)$ satisfies the following conditions:

H-1: $f(t, u)$ is continuous in D: $t \geqq 0$, $-\infty < u < \infty$.
H-2: The derivative f_u exists on D and satisfies $f_u(t, u) > 0$ on D.
H-3: $|f(t, u(t))| \leqq f_u(t, 0)|u(t)|$ on D.

An important class of functions $f(t, u)$ which satisfy conditions H-1, 2, 3 is the class of twice continuously differentiable functions $f(t, u)$ which are odd and strictly monotone in u with $f_{uu} \geqq 0$ for $u < 0$ and $f_{uu} \leqq 0$ for $u > 0$. Nonlinear eigenvalue problems involving this class of functions have been studied extensively by G. H. Pimbley [**1**].

For the case $f(t, u) = \pm t^\sigma u^n$, R. Bellman [**2**] has given an exhaustive treatment of the asymptotic behavior of proper solutions (i.e., solutions which exist and have continuous derivatives for $t \geqq t_0$). For the case $f(t, u) = a(t)u^{2n+1}$ several results on asymptotic behavior exist depending on properties of $a(t)$. References can be found in the papers of P. Waltman [**3**] and R. A. Moore and Z. Nehari [**4**].

Our basic result is that there exist solutions of (1) which approach those of $u'' = 0$. More precisely, we prove the

Received by the editors October 5, 1966.

[1] This work was supported by the National Science Foundation under Grant No. GP-4597 at the California Institute of Technology.