DIFFERENTIAL SYSTEMS ON FIBERED MANIFOLDS

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Let $M$ and $M'$ be real analytic manifolds and $\rho: M \rightarrow M'$ an analytic map which is surjective and whose rank is equal to the dimension of $M'$ at every point of $M$. We shall denote respectively by $\rho_*$ and $\rho^*$ the maps induced by $\rho$ on tangent vectors and differential forms.

Let $\Sigma$ and $\Sigma'$ be analytic exterior differential systems defined respectively in $M$ and $M'$ and assume that for every differential form $\omega \in \Sigma'$, $\rho^* \omega$ belongs to $\Sigma$. Take a point $x_0 \in M$ and put $x'_0 = \rho(x_0)$. Let $\mathcal{U}'p$ be an integral manifold of $\Sigma'$ going through $x'_0$. In this note we give a condition for the existence of an integral manifold $\mathcal{U}p$ of $\Sigma$ going through $x_0$ such that, for a suitable neighborhood $U$ of $x_0$, $\rho(\mathcal{U}p) = \mathcal{U}'p \cap \rho(U)$.

The proof consists in a careful application to our situation of the technic of the Cartan-Kähler theory. The situation we study here appears in the theory of continuous pseudogroups (see [3, p. 125]). For the definitions and results we use of the theory of exterior differential systems we refer the reader to [1] and [2].

For any integral contact element $E^k(x_0)$ of $\Sigma$ denote by $J(E^k(x_0))$ the polar space of $E^k(x_0)$ and by $J'(E^k(x_0))$ the subspace of $J(E^k(x_0))$ of all forms $\omega \wedge X_1 \wedge \cdots \wedge X_r$, where $\omega$ is a form of degree $r+1$ belonging to $\rho^*(\Sigma')$ and $X_1, \cdots, X_r$ are vectors of $E^k(x_0)$.

Let $F_{x_0}$ be the tangent space to the fiber of $M$ at the point $x_0$ and denote by $J(E^k(x_0)) \mid F_{x_0}$ the space obtained restricting the forms of $J(E^k(x_0))$ to the subspace $F_{x_0}$.

Denote by $E'p(x'_0)$ the tangent space of $\mathcal{U}'p$ at $x'_0$ and assume that there exists an ordinary integral element $E^p(x_0)$ of $\Sigma$ such that $\rho_*(E^p(x_0)) = E'p(x_0)$. Assume moreover there exists a sequence $E^0(x_0) \subset E^1(x_0) \subset \cdots \subset E^{p-1}(x_0)$ of regular contact elements contained in $E^p(x_0)$ and such that $\dim J(E^k(x_0)) - \dim J'(E^k(x_0)) = \dim J(E^k(x_0)) \mid F_{x_0}$, $0 \leq k \leq p - 1$. Under these assumptions we shall prove the following theorem.

**Theorem.** There exists an integral manifold $\mathcal{U}p$ of $\Sigma$ defined in a neighborhood $U$ of $x_0$ such that the tangent space $\mathcal{U}'p \mid x_0$ of $\mathcal{U}p$ at the point $x_0$ is $E^p(x_0)$ and $\rho(\mathcal{U}p) = \mathcal{U}'p \cap \rho(U)$.

**Proof.** Choose coordinates $x^i$, $1 \leq i \leq n'$ in $M'$, defined in a neighborhood of $x'_0$ and coordinates $x^i$, $y^j$, $1 \leq j \leq n$ in $M$, defined in a

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neighborhood of \( x_0 \) such that \( dx^{k+1} | E^k(x_0) = \cdots = dx^n | E^k(x_0) = dy^j | E^k(x_0) = 0, \ 1 \leq k \leq p, 1 \leq j \leq n \) (we shall use the same notation for \( x^i \) and \( x^i \circ \rho \)). Since the differentials \( dx^1, \cdots, dx^p \) are linearly independent on \( E^p(x_0) \), \( U^p \) is locally defined by equations \( x^\lambda = H^\lambda(x^1, \cdots, x^p), p + 1 \leq \lambda \leq n' \). Since \( U^p \) is tangent to \( E^p(x'_0) \) and \( dx^\lambda | E^p(x_0) = 0, (\partial H^\lambda / \partial x^i)_{x_0} = 0 \). Let \( V^1 \) be the curve in \( M' \) defined by equations \( x^2 = \cdots = x^p = 0, x^\lambda = H^\lambda(x^1, 0, \cdots, 0), p + 1 \leq \lambda \leq n' \), and put \( E^1(x_0) = \rho^* (E^1(x_0)) \). Clearly \( U^1_0 = E^1(x_0) \). We want to show that there is an integral curve of \( \Sigma \) which covers \( U^1 \). Let \( \tilde{\eta}^1, \cdots, \tilde{\eta}^p \) be a basis of the space of forms of degree 1 of \( \Sigma' \) at the point \( x'_0 \) and choose forms \( \xi^1, \cdots, \xi^p \) such that \( \tilde{\eta}^1, \cdots, \tilde{\eta}^p, \xi^1, \cdots, \xi^p \) is a basis of the forms of degree 1 of \( \Sigma \) at the point \( x_0 \) (we denote the form \( \rho^* \tilde{\eta}^i \) also by \( \tilde{\eta}^i \)). Since \( x_0 \) is a regular point of \( \Sigma \) there are forms \( \eta^1, \cdots, \eta^{p+1}, \xi^1, \cdots, \xi^p \) defined in a neighborhood of \( x_0 \) such that they are a basis of the space of 1-forms of \( \Sigma \) in this neighborhood and \( \eta^i_{x_0} = \tilde{\eta}^i, \xi^i_{x_0} = \xi^i \). Put

\[
\begin{align*}
(1) \quad \eta^\alpha = \sum_{i=1}^{n'} A^\alpha_i(x) dx^i, \quad 1 \leq \alpha \leq \alpha_1, \\
(2) \quad \xi^\beta = \sum_{i=1}^{n'} B^\beta_i(x, y) dx^i + \sum_{j=1}^{n} C^\beta_j(x, y) dy^j, \quad 1 \leq \beta \leq \beta_1.
\end{align*}
\]

From the hypothesis, the matrix \( ||C^\beta_j(x, y)|| \) has maximum rank at the point \( x_0 \). Hence, the linear equations

\[
(3) \quad B^\beta_i(x, y) + \sum_{\lambda=p+1}^{s+1} B^\beta_\lambda \frac{\partial H^\lambda}{\partial x^1} + \sum_{j=1}^{n} C^\beta_j(x, y) \frac{dy^j}{dx^1} = 0
\]

can be solved with respect to some of the variables \( dy^i / dx^i \). Assume that they can be solved with respect to \( dy^1 / dx^1, \cdots, dy^n / dx^1 \). In (3), put, \( x^2 = \cdots = x^{n'} = 0 \) and replace the variables \( y^{s+1}, \cdots, y^n \) by arbitrary functions \( F^{s+1}(x^1), \cdots, F^n(x^1) \) such that \( (d F^k / dx^1) = 0, s+1 \leq k \leq n \). Let \( F^k(x^1), 1 \leq k \leq s \), be the solution of the resulting system with the initial conditions \( (d F^k / dx^1)_{x^1=0} = 0, 1 \leq k \leq s \). Then the curve \( x^2 = \cdots = x^p = 0, x^\lambda = H^\lambda(x^1, 0, \cdots, 0), y^j = F^j(x^1), p + 1 \leq \lambda \leq n', 1 \leq j \leq n \), is an integral curve of \( \Sigma \) which covers \( U^1 \).

Assume now, by induction, that we have lifted the manifold \( U^{r-1} \) defined by the equations \( x^r = \cdots = x^p = 0, x^\lambda = H^\lambda(x^1, \cdots, x^{r-1}, 0, \cdots, 0) \) to an integral manifold \( U^{r-1} \) of \( \Sigma \) which is tangent \( E^{r-1}(x_0) \). Let \( \tilde{\eta}^1, \cdots, \tilde{\eta}^n, \xi^1, \cdots, \xi^p \) be a basis of \( J(E^{r-1}(x_0)) \) such that \( \tilde{\eta}^1, \cdots, \tilde{\eta}^n, \xi^1, \cdots, \xi^p \) is a basis of \( J'(E^{r-1}(x_0)) \). For a contact element \( E^r(x) \) sufficiently close to \( E^r(x_0) \), we can write
\[ dx^\lambda \big| E^r(x) = \sum_{i=1}^{r} u_i^\lambda dx^i, \quad r + 1 \leq \lambda \leq n', \]
\[ dy^\lambda \big| E^r(x) = \sum_{i=1}^{n'} v_i^\lambda dx^i, \quad 1 \leq j \leq n. \]

Put
\[ L_i(E^r(x)) = \frac{\partial}{\partial x^i} + \sum_{\lambda=r+1}^{n'} u_i^\lambda \frac{\partial}{\partial x^\lambda} + \sum_{j=1}^{n} v_i^j \frac{\partial}{\partial y^j}, \quad 1 \leq i \leq r, \]
and assume that
\[ \tilde{\eta}^\alpha = (L_i(E^r(x_0)) \wedge \cdots \wedge L_{i-1}(E^r(x_0)))_\omega, \quad 1 \leq \alpha \leq \alpha_r-1, \]
where \( \omega \) is a form of degree \( t \) in \( \rho^*\Sigma' \). For \( E^r(x) \) sufficiently close to \( E^r(x_0) \) define
\[ \eta^\alpha(E^r(x)) = (L_i(E^r(x)) \wedge \cdots \wedge L_{i-1}(E^r(x)))_\omega. \]

Define forms \( \xi^\beta \) in a similar way. Let
\[ \eta^\alpha(E^r(x)) = \sum_{i=1}^{n'} A_i^\alpha(x, u) dx^i, \]
\[ \xi^\beta(E^r(x)) = \sum_{i=1}^{n'} B_i^\beta(x, y, u, v) dx^i + \sum_{j=1}^{n} C_j^\beta(x, y, u, v) dy^j, \]
be the expression of these forms in local coordinates. Observe that the coefficients are functions only of the variables \( x, y, u^i, v^j \) with \( 1 \leq i \leq r-1 \).

By hypothesis the matrix \( ||C^\beta_j|| \) has maximum rank at the point \( E^r(x_0) \). Assume that the equations of \( \Upsilon^{r-1} \) are \( x^r = \cdots = x^p = 0, \]
\( x^\lambda = H^\lambda(x^1, \cdots, x^{r-1}, 0, \cdots, 0), \]
\( y^j = F^j(x^1, \cdots, x^r) \) and construct the functions
\[ \eta^\alpha(E^r(x))(L_r(E^r(x))) = A_r^\alpha + \sum_{\lambda=r+1}^{n'} A^\alpha_\lambda u^\lambda, \]
\[ \xi^\beta(E^r(x))(L_r(E^r(x))) = B_r^\beta + \sum_{\lambda=r+1}^{n'} B^\beta_\lambda u^\lambda + \sum_{j=1}^{n} C^\beta_j v^j. \]

Consider the following system of partial differential equations, obtained replacing in the above functions \( u^\lambda_i \) by \( \partial x^\lambda / \partial x^i \) and \( v^j_i \) by \( \partial y^j / \partial x^i \):
\[ \begin{aligned}
(4) & \quad A_r^\alpha \left( x, \frac{\partial x^\lambda}{\partial x^i} \right) + \sum_{\lambda=r+1}^{n'} A_{\lambda}^\alpha \left( x, \frac{\partial x^\lambda}{\partial x^i} \right) \frac{\partial x^\lambda}{\partial x^r} = 0, \\
& \quad B_r^\beta \left( x, y, \frac{\partial x^\lambda}{\partial x^i}, \frac{\partial y^j}{\partial x^i} \right) + \sum_{\lambda=r+1}^{n'} B_{\lambda}^\beta \left( x, y, \frac{\partial x^\lambda}{\partial x^i}, \frac{\partial y^j}{\partial x^i} \right) \frac{\partial x^\lambda}{\partial x^r} \\
& \quad + \sum_{j=1}^{n} C_j^\gamma \left( x, y, \frac{\partial x^\lambda}{\partial x^i}, \frac{\partial y^j}{\partial x^i} \right) \frac{\partial y^j}{\partial x^r} = 0.
\end{aligned} \]

We want to lift the manifold \( \mathcal{U} \) defined by \( x_r+1 = \cdots = x_p = 0 \) \( x^\lambda = H^\lambda(x^1, \cdots, x_r, 0, \cdots, 0) \) to an integral manifold of \( \Sigma \). Observe that the functions \( H^\lambda(x^1, \cdots, x_r, 0, \cdots, 0) \) are solutions of equations (4). Assume that the equations of \( \mathcal{U} \) are \( x_r = \cdots = x_p = 0 \), \( x^\lambda = H^\lambda(x_1, \cdots, x_{r-1}, 0, \cdots, 0) \), \( y^j = G^j(x_1, \cdots, x_{r-1}) \). Equations (5) can be solved with respect to some of the variables \( \partial y^j/\partial x^r \); assume that they can be solved with respect to \( \partial y^1/\partial x^r, \cdots, \partial y^s/\partial x^r \).

Put in (5),
\[ \frac{\partial x^r+1}{\partial x^r} = \cdots = \frac{\partial x^p}{\partial x^r} = 0, \quad \frac{\partial x^\lambda}{\partial x^r} = \frac{\partial H^\lambda(x^1, \cdots, x_r)}{\partial x^r}, \]
\[ \rho + 1 \leq \lambda \leq n', \]
and replace the variables \( y^j \) by arbitrary functions \( y^j = F^j(x^1, \cdots, x_r) \), \( s+1 \leq j \leq n \), subjected to the restrictions \( \partial F^j/\partial x^i)(0, \cdots, 0) = 0 \) and \( F^j(x^1, \cdots, x_{r-1}, 0) = G^j(x^1, \cdots, x_{r-1}), 1 \leq i \leq r, s+1 \leq j \leq n \). Let \( y^j = F^j(x^1, \cdots, x_r), 1 \leq j \leq s \), be the solution of the resulting Cauchy-Kowaleswky system with the initial conditions \( F^j(x^1, \cdots, x_{r-1}, 0) = G^j(x^1, \cdots, x_{r-1}), 1 \leq j \leq s \). Then, as in the Cartan-Kähler theorem, the manifold \( \mathcal{U} \) defined by the equations \( x_r+1 = \cdots = x_p = 0, \)
\( x^\lambda = H^\lambda(x^1, \cdots, x_r), y^j = F^j(x^1, \cdots, x_r), 1 \leq j \leq n \), is a solution of \( \Sigma \) which covers \( \mathcal{U} \). The Theorem is proved.

**References**


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