THE MOD 2 HOMOLOGY OF THE IMAGE OF AN EXACTLY 2 TO 1 MAP FROM A SPHERE

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The real projective space and the space formed by pinching together the north and south poles of a 2 sphere, $S^2$, are both the image of $S^2$ under an exactly 2 to 1 map. (A map is said to be exactly 2 to 1 if it is onto, continuous, and the inverse of each point is exactly two points.) Clearly these spaces are not homeomorphic; however, from the standpoint of additive mod 2 homology, they are indistinguishable. In fact, from the mod 2 standpoint there is only one type of image.\(^1\) Precisely,

**Theorem.** Suppose $f: Y \rightarrow X$ is a 2-1 and $Y$ is a mod 2 homology sphere. Then $H_i(X, \mathbb{Z}_2)$ is $\mathbb{Z}_2$ if $0 \leq i \leq \phi(Y)$, and 0 otherwise.

Here we shall denote the homology dimension of a space $Y$ by $\phi(Y)$ and abbreviate “exactly 2 to 1 map” by 2-1.

Briefly, one could say if there exists a 2-1 from an $n$-sphere onto a space, then that space has the mod 2 homology of the $n$-dimensional projective space.

One easy application of the theorem is the

**Lemma.** The circle is the only sphere which can be mapped by 2-1 onto a sphere.

The theorem is proved using a characterization of 2-1 described by Černavskij [1] and the author [2] which generalize the works of P. Civin [3]. Inserted before the proof will be a brief summary of these works and some pertinent facts from Smith theory. All spaces are assumed to be Hausdorff and the homology group to be Čech with the integers mod 2 as coefficients.

Suppose that $f: Y \rightarrow X$ is 2-1 and $Y$ is a mod 2 homology manifold, then there exists an open set $U \subset Y$ together with an involution $T$ on $Y$ such that $T$ moves a point on $Y$ if and only if it is in $U$. (A homeomorphism $R$ is an involution if $R \circ R$ is the identity.) Moreover, taking $S_1 = Y - U$, i.e., the fixed point set of $T$, and $f(S_1) = Z$, then $f$ factors as shown in the following diagram:

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In addition:

(i) \( f = k \circ h \),

(ii) \( h \) is the map from \( Y \) to the decomposition or orbit space of \( T \),

(iii) \( h|S_1 \) is a homeomorphism,

(iv) \( k| (Y/T - h(S_1)) \) is a homeomorphism,

(v) \( k| h(S_1) \) is 2-1,

(vi) \( S_1 \) is the disjoint union of mod 2 homology manifolds.

Facts (i) through (v) follow from Civin's construction, and (vi) by P. A. Smith [4]. By specializing to the case that \( Y \) is a mod 2 homology sphere, it follows that \( S_1 \) is one of lower dimension. Repeated use of this fact along with the above factorization yields a sequence \( S_0, S_1, \cdots, S_n \) of mod 2 homology spheres along with involutions \( T_i \) on \( S_i \) with:

(vii) \( S_0 = Y \) and \( S_n = \emptyset \) and as above

\[
S_i \xrightarrow{k_i} S_i/T_i \xrightarrow{k_i} X
\]

(viii)

\[
S_{i+1} \rightarrow h_i(S_{i+1}) \rightarrow f(S_{i+1})
\]

An examination of the Civin construction shows that it is unique and gives only one decomposition for each 2-1.

One additional fact from Smith theory will be used. In the case \( Y \) is an homology \( n \)-sphere, then the orbit map \( h: Y \rightarrow Y/T \) induces a trivial map \( h_*: H_n(Y) \rightarrow H_n(Y/T) \).

**Proof of the Theorem.** The proof proceeds by induction on \( n \) the number of terms in the decomposition for \( f \). If \( n = 1 \), then \( f \) is the orbit map for a fixed point free involution on \( X \) and the theorem follows by Smith theory. Now suppose that the theorem is true for maps whose decomposition has \( N \) or less terms and further assume that \( n = N+1 \). Referring to the first diagram, \( f|S_1 \) is 2-1 and this map has a decomposition with \( N \) terms so that:

(ix) \( H_i(Z) = \mathbb{Z}_2 \) if \( 0 \leq i \leq \phi(S_1) \) and 0 otherwise. Now \( H_i(X, Z) \approx H_i(S_0/T, S_1) \) by \( k_* \), (iv), and the continuity of Čech homology. Moreover, \( H_i(S_0/T, S_1) \) is \( \mathbb{Z}_2 \) if \( \phi(S_1) < i \leq \phi(S_0) \) and 0 otherwise, again by Smith theory [Remark 2, p. 163 of 4].

**Calculation of \( H_q(X) \).**

**Case 1.** \( q > \phi(S_1) + 1 \).
Using the exact sequence for the pair \((X, Z)\)

\[ 0 \cong H_q(Z) \to H_q(X) \to H_q(X, Z) \to H_{q-1}(Z) \cong 0. \]

The end terms are 0 by (vii), so \(H_q(X) \cong H_q(X, Y)\). As the latter is isomorphic to \(H_q(S_0/T, S_1)\), then \(H_q(X)\) is \(\mathbb{Z}_2\) if \(q < \phi(S_0)\) and 0 otherwise.

**Case II.** \(q = \phi(S_1) + 1\).

The homeomorphism induced on the exact sequence of the pairs \((Y/T, S_1)\) and \((X, Z)\) yield in part

\[
\begin{align*}
0 & \cong H_q(Z) \to H_q(X) \to H_q(X, Z) \to H_{q-1}(Z) \\
& \cong H_q(Y/T, S_1) \to H_q(S_1) \\
& \cong H_{q-1}(S_1).
\end{align*}
\]

Now \(H_q(Y/T, S_1)\) is \(\mathbb{Z}_2\), and \(g^*\) an isomorphism. Granting that \(g|_{S_1*}\) is trivial it follows that \(\beta\) is also. Thus as \(H_q(Z) \cong 0\), \(\alpha\) is an isomorphism and \(H_q(X) = \mathbb{Z}_2\). The triviality of \(g|_{S_1}\) comes directly from Smith theory if \(n = 2\) in which case \(g|_{S_1}\) is the decomposition map of an involution. On the other hand, if \(n > 2\) then as \(g|_{S_1}\) is 2-1 to \(Z\), \(g|_{S_1} = k_1 \circ h_1\) where \(h_1\) is a decomposition map of an involution on \(S_1\) and so \(k_1*\) is trivial. Thus \(g|_{S_1*}\) is trivial.

**Case III.** \(q = \phi(S_1)\).

Again referring to part of the sequence for the pair \((X, Y)\)

\[
0 \cong H_{q+1}(Z) \to H_{q+1}(X) \to H_{q+1}(X, Z) \to H_q(Z) \to H_q(X) \to H_q(Y/T, S_1) = 0.
\]

Both \(H_{q+1}(X)\) and \(H_{q+1}(X, Z)\) are \(\mathbb{Z}_2\), thus \(\partial\) has image 0 and \(H_q(X) \cong H_q(Z)\). But \(H_q(Z) \cong \mathbb{Z}_2\) by induction.

**Case IV.** \(q < \phi(S_1)\).

Finally:

\[
0 \cong H_{q+1}(X, Z) \to H_n(Z) \to H_n(X) \to H_n(X, Z) \cong 0.
\]

So \(H_q(X) \cong H_q(X)\), which is \(\mathbb{Z}_2\) by (ix).

**Corollary.** There is no 2-1 from \(S^i\) to \(S^j\) unless \(i = j = 1\).

**Proof.** If \(i > 1\) then \(H_i(S^j) = H_{i-1} - (S^j) = Z\) by the theorem which is a contradiction. In the case \(i = j = 1\) two 2-1 maps are well known.

**Question.** Which manifolds have the property that all images of 2-1 have the same mod 2 homology groups? An example of a manifold which does not possess this property is \(S^1 \times S^2\). Taking \(T_i\) to be a
fixed point free involution on \( S^i \) and the identity on the other component, it is easy to see that:

\[
H_1\left(\frac{S^1 \times S^2}{T_1}, Z_2\right) \neq H_1\left(\frac{S_1 \times S_2}{T_2}, Z_2\right).
\]

**Bibliography**


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**VON NEUMANN REGULARITY IN JORDAN ALGEBRAS**

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In a Jordan algebra \( J \) (not necessarily finite-dimensional), an element \( a \) is regular if there exists an element \( x \) in \( J \) such that \( xU_a = 2(a \cdot x) \cdot a - a^2 \cdot x = a \). The algebra is a regular algebra if every element in \( J \) is regular (see [1, p. 246]). If \( J = A^+ \), where \( A \) is an associative algebra, then \( a \) is regular in \( J \) if, and only if, it is von Neumann regular in the associative algebra \( A \); that is if there exists an element \( x \) in \( A \) such that \( a \cdot x \cdot a = a \). The purpose of this note is to carry out some analogous results for Jordan algebras (not necessarily special Jordan algebras). The characteristic of the ground field \( K \) is always assumed not to be two.

Throughout this paper, the following three identities will be used quite frequently.

\[
\begin{align*}
(x \cdot y) \cdot d & \cdot z + [(x \cdot z) \cdot d] \cdot y + [(y \cdot z) \cdot d] \cdot x = (x \cdot y) \cdot (d \cdot z) \\
& + (x \cdot z) \cdot (d \cdot y) + (y \cdot z) \cdot (d \cdot x) \quad \text{for all } x, y, z, d \text{ in } J.
\end{align*}
\]

(2) \( U_xU_y = U_yU_xU_y \) for all \( x, y \) in \( J \).

(3) \( U_xU_{a-x} = U_aU_xU_a + U_a - 2L(a \cdot x)U_a - 2L(a)L(x)U_a \\
+ 2L(x)L(a)U_a \quad \text{for all } a, x \text{ in } J.\)

Identity (1) is the linearized form of the Jordan identity \( x^2 \cdot (x \cdot y) = x \cdot (x^2 \cdot y) \). Identity (2), which is called the fundamental formula of Jordan algebras, was proved in several places, (see, for example, [2]).

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