THE MOD 2 HOMOLOGY OF THE IMAGE OF AN EXACTLY 2 TO 1 MAP FROM A SPHERE

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The real projective space and the space formed by pinching together the north and south poles of a 2 sphere, S^2, are both the image of S^2 under an exactly 2 to 1 map. (A map is said to be exactly 2 to 1 if it is onto, continuous, and the inverse of each point is exactly two points.) Clearly these spaces are not homeomorphic; however, from the standpoint of additive mod 2 homology, they are indistinguishable. In fact, from the mod 2 standpoint there is only one type of image.[1] Precisely,

THEOREM. *Suppose* f: $Y \to X$ *is a* 2-1 *and* Y *is a* mod 2 *homology sphere. Then* $H_i(X, Z_2)$ *is* Z_2 *if* $0 \leqq i \leqq \phi(Y)$, *and* 0 *otherwise.*

Here we shall denote the homology dimension of a space Y by $\phi(Y)$ and abbreviate "exactly 2 to 1 map" by 2-1.

Briefly, one could say if there exists a 2-1 from an n-sphere onto a space, then that space has the mod 2 homology of the n-dimensional projective space.

One easy application of the theorem is the

LEMMA. *The circle is the only sphere which can be mapped by* 2-1 *onto a sphere.*

The theorem is proved using a characterization of 2-1 described by Černavskiĭ [1] and the author [2] which generalize the works of P. Civin [3]. Inserted before the proof will be a brief summary of these works and some pertinent facts from Smith theory. All spaces are assumed to be Hausdorff and the homology group to be Čech with the integers mod 2 as coefficients.

Suppose that f: $Y \to X$ is 2-1 and Y is a mod 2 homology manifold, then there exists an open set $U \subset Y$ together with an involution T on Y such that T moves a point on Y if and only if it is in U. (A homeomorphism R is an involution if $R \circ R$ is the identity.) Moreover, taking $S_1 = Y - U$, i.e., the fixed point set of T, and $f(S_1) = Z$, then f factors as shown in the following diagram:

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[1] The author gratefully acknowledges various helpful criticisms and suggestions made by the referee.

$$\begin{array}{ccccc} Y & \xrightarrow{h} & Y/T & \xrightarrow{k} & X \\ \cup & & \cup & & \cup \\ S_1 & \xrightarrow{h|S_1} & h(S_1) & \xrightarrow{k|h(S_1)} & Z \end{array}$$

In addition:

(i) $f = k \circ h$,

(ii) h is the map from Y to the decomposition or orbit space of T,

(iii) $h|S$ is a homeomorphism,

(iv) $k|(Y/T - h(S_1))$ is a homeomorphism,

(v) $k|h(S_1)$ is 2-1,

(vi) S_1 is the disjoint union of mod 2 homology manifolds.

Facts (i) through (v) follow from Civin's construction, and (vi) by P. A. Smith [**4**]. By specializing to the case that Y is a mod 2 homology sphere, it follows that S_1 is one of lower dimension. Repeated use of this fact along with the above factorization yields a sequence $S_0, S_1, \cdots, S_n$ of mod 2 homology spheres along with involutions T_i on S_i with:

(vii) $S_0 = Y$ and $S_n = \varnothing$ and as above

$$\text{(viii)} \qquad \begin{array}{ccccc} S_i & \xrightarrow{k_i} & S_i/T_i & \xrightarrow{k_i} & X \\ \cup & & \cup & & \cup \\ S_{i+1} & \longrightarrow & h_i(S_{i+1}) & \longrightarrow & f(S_{i+1}) \end{array}$$

An examination of the Civin construction shows that it is unique and gives only one decomposition for each 2-1.

One additional fact from Smith theory will be used. In the case Y s an homology n-sphere, then the orbit map $h: Y \to Y/T$ induces a trivial map $h_*: H_n(Y) \to H_n(Y/T)$.

Proof of the Theorem. The proof proceeds by induction on n the number of terms in the decomposition for f. If $n = 1$, then f is the orbit map for a fixed point free involution on X and the theorem follows by Smith theory. Now suppose that the theorem is true for maps whose decomposition has N or less terms and further assume that $n = N + 1$. Referring to the first diagram, $f|S_1$ is 2-1 and this map has a decomposition with N terms so that:

(ix) $H_i(Z)$ is Z_2 if $0 \leqq i \leqq \phi(S_1)$ and 0 otherwise. Now $H_i(X, Z) \approx H_i(S_0/T, S_1)$ by k_*, (iv), and the continuity of Čech homology. Moreover, $H_i(S_0/T, S_1)$ is Z_2 if $\phi(S_1) < i \leqq \phi(S_0)$ and 0 otherwise, again by Smith theory [Remark 2, p. 163 of **4**].

Calculation of $H_q(X)$.

Case 1. $q > \phi(S_1) + 1$.

Using the exact sequence for the pair (X, Z)

$$0 \approx H_q(Z) \to H_q(X) \to H_q(X, Z) \to H_{q-1}(Z) \approx 0.$$

The end terms are 0 by (vii), so $H_q(X) \approx H_q(X, Y)$. As the latter is isomorphic to $H_q(S_0/T, S_1)$, then $H_q(X)$ is Z_2 if $q < \phi(S_0)$ and 0 otherwise.

Case II. $q = \phi(S_1) + 1$.

The homeomorphism induced on the exact sequence of the pairs $(Y/T, S_1)$ and (X, Z) yield in part

$$\begin{array}{ccccccc} & & & & H_q(Y/T, S_1) & \to & H_{q-1}(S_1) \\ & & & & \downarrow g'_* & & \downarrow g|S_{1_*} \\ 0 \approx & H_q(Z) & \to & H_q(X) & \xrightarrow{\alpha} H_q(X, Z) & \xrightarrow{\beta} & H_{q-1}(Z) \end{array}$$

Now $H_q(Y/T, S_1)$ is Z_2, and g'_* an isomorphism. Granting that $g|S_{1_*}$ is trivial it follows that β is also. Thus as $H_q(Z) \approx 0$, α is an isomorphism and $H_q(X) = Z_2$. The triviality of $g|S_{1_*}$ comes directly from Smith theory if $n = 2$ in which case $g|S_1$ is the decomposition map of an involution. On the other hand, if $n > 2$ then as $g|S_1$ is 2-1 to Z, $g|S_1 = k_1 \circ h_1$ where h_1 is a decomposition map of an involution on S_1 and so k_{1_*} is trivial. Thus $g|S_{1_*}$ is trivial.

Case III. $q = \phi(S_1)$.

Again referring to part of the sequence for the pair (X, Y)

$$\begin{aligned} 0 \approx H_{q+1}(Z) \to H_{q+1}(X) \to H_{q+1}(X, Z) \xrightarrow{\partial_*} H_q(Z) \to H_q(X) \\ \to H_q(X, Z) \to H_q(Y/T, S_1) = 0. \end{aligned}$$

Both $H_{q+1}(X)$ and $H_{q+1}(X, Z)$ are Z_2, thus ∂_* has image 0 and $H_q(X) \approx H_q(Z)$. But $H_q(Z) \approx Z_2$ by induction.

Case IV. $q < \phi(S_1)$.

Finally:

$$0 \approx H_{q+1}(X, Z) \to H_n(Z) \to H_n(X) \to H_n(X, Z) \approx 0.$$

So $H_q(X) \approx H_q(X)$, which is Z_2 by (ix).

COROLLARY. *There is no* 2-1 *from* S^i *to* S^j *unless* $i = j = 1$.

PROOF. If $i > 1$ then $H_i(S^j) = H_{i-1} - (S^j) = Z$ by the theorem which is a contradiction. In the case $i = j = 1$ two 2-1 maps are well known.

QUESTION. *Which manifolds have the property that all images of* 2-1 *have the same* mod 2 *homology groups*? An example of a manifold which does not possess this property is $S^1 \times S^2$. Taking T_i to be a

fixed point free involution on S^i and the identity on the other component, it is easy to see that:

$$H_1\left(\frac{S^1 \times S^2}{T_1}, Z_2\right) \neq H_1\left(\frac{S_1 \times S_2}{T_2}, Z_2\right).$$

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VON NEUMANN REGULARITY IN JORDAN ALGEBRAS

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In a Jordan algebra J (not necessarily finite-dimensional), an element a is regular if there exists an element x in J such that $xU_a = 2(a\cdot x)\cdot a - a^2\cdot x = a$. The algebra is a regular algebra if every element in J is regular (see [**1**, p. 246]). If $J = A^+$, where A is an associative algebra, then a is regular in J if, and only if, it is von Neumann regular in the associative algebra A; that is if there exists an element x in A such that $axa = a$. The purpose of this note is to carry out some analogous results for Jordan algebras (not necessarily special Jordan algebras). The characteristic of the ground field K is always assumed not to be two.

Throughout this paper, the following three identities will be used quite frequently.

$$\begin{aligned}(1)\qquad &[(x\cdot y)\cdot d]\cdot z + [(x\cdot z)\cdot d]\cdot y + [(y\cdot z)\cdot d]\cdot x = (x\cdot y)\cdot(d\cdot z)\\ &\quad + (x\cdot z)\cdot(d\cdot y) + (y\cdot z)\cdot(d\cdot x) \qquad \text{for all } x, y, z, d \text{ in } J.\end{aligned}$$

$$(2)\qquad U_{xU_y} = U_yU_xU_y \qquad \text{for all } x, y \text{ in } J.$$

$$\begin{aligned}(3)\qquad U_{xU_a-a} &= U_aU_xU_a + U_a - 2L(a\cdot x)U_a - 2L(a)L(x)Ua\\ &\quad + 2L(x)L(a)Ua \qquad \text{for all } a, x \text{ in } J.\end{aligned}$$

Identity (1) is the linearized form of the Jordan identity $x^2\cdot(x\cdot y) = x\cdot(x^2\cdot y)$. Identity (2), which is called the fundamental formula of Jordan algebras, was proved in several places, (see, for example, [**2**]).

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