SOME LINEAR TOPOLOGICAL PROPERTIES OF
SEPARABLE FUNCTION ALGEBRAS

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1. Introduction. Let \( C(S) \) be the Banach algebra of all continuous
complex-valued functions on a compact Hausdorff space \( S \) with the
norm \( \|f\| = \sup_{s \in S} |f(s)| \).

By a function algebra we mean a closed subalgebra of \( C(S) \) which
contains constant functions and separates the points of \( S \).

In this paper we study some linear topological properties of separa-
ble (= \( S \) metrizable) function algebras. In other words, we are inter-
ested in the properties of a Banach space which is either linearly
homeomorphic or linearly isometric\(^3\) to a function algebra on a com-
pact metric space. If \( S \) is countable, then \( C(S) \) is the unique function
algebra on \( S \) [16]. Therefore for \( S \) metrizable the only interesting
case is when \( S \) is uncountable.

Our main result (Theorem 1) shows that in this case the function
algebras still possess several properties which are possessed by the
space \( C(S) \). In particular, if \( A \) is a function algebra on a compact
uncountable metric space \( S \), then every separable normed linear space
is isometric to a closed linear subspace of \( A \).

2. Notation. \( M(S) \), \( S \) compact Hausdorff will denote the usual
Banach space of all complex finite regular Borel measures on \( S \), the
dual of the space \( C(S) \). For \( \mu \) in \( M(S) \) we shall employ the notation
\( \mu(f) = \int_{S} f \, d\mu \). The dual space to a normed linear space \( X \) will be de-
noted by \( X^* \).

A function \( g \) in \( C(S) \) peaks on the subset \( T \subset S \) provided
\( g^{-1}(\|g\|) = T \) and \( |g(s)| < \|g\| \) for \( s \in S \setminus T \). If \( T = \{t\} \) is a one-point set,
we shall say that \( g \) peaks at \( t \). If \( A \) is a function algebra on a compact
metric space \( S \), then the set

\[ B(A) = \{ s \in S : \text{there is some } g \in A \text{ such that } g \text{ peaks at } s \} \]
is called the Choquet boundary for $A$ (Bishop [1], cf. also [15, p. 53]). The set $B(A)$ is an absolute $G_δ$, i.e., it is a $G_δ$ in a compact metric space $S$.

A (nonempty) subset $S_0$ of a topological space $S$ is said to be a (proper) perfect set if $S_0$ is closed and if it is dense in itself, i.e., every point of $S_0$ is a limit point of $S_0$. Finally if $T$ is a subset of a topological space $S$, then $\text{int } T$ denotes the interior of $T$; if $S$ is metric, then $d(s', s'')$ denotes the distance between points $s'$ and $s''$ in $S$.

3. “Peano curves” in function algebras. In this section we shall show that if $S$ is uncountable, then in any function algebra on $S$ there are functions whose range is the whole unit disc

$$K = \{ z : |z| \leq 1 \}$$

(like the Peano curve which maps the unit interval onto the unit square).

PROPOSITION 1. Let $A$ be a function algebra on a compact metric space $S$ and let $S_0$ be a proper perfect compact subset of $B(A)$. Then there is a function $h$ in $A$ such that $h(S) = K$ and $h(S \setminus S_0) \subset \text{int } K$.

PROOF. Since $S \setminus S_0$ is an $F_\sigma$, there is an increasing sequence of closed sets $F_1 \subset F_2 \subset \cdots$ such that $S \setminus S_0 = \bigcup_{n=1}^{\infty} F_n$. Let us set

$$K_n = \{ z \in K : |z| \leq 1 - 2^{-n} \} \quad \text{for } n = 1, 2, \cdots.$$ We shall define by induction a sequence $(f_n)$ in $A$ satisfying the following properties.

(1) $\|f_n\| < 1$,

(2) there is a finite set $Z_n$ in $S_0$ such that $f_n(Z_n) \subset \text{int } K_n$ and $f_n(Z_n)$ is a $2^{-n}$ net for $K_{n+1}$,

(3) $\|f_{n+1} - f_n\| < 2^{-n}$,

(4) if $s \in F_m$, then $|f_n(s)| < \frac{1}{2}(1 + \|f_{m+1}\|)$ (1 $\leq m \leq n$; $n = 1, 2, \cdots$).

Let us set $f_0 = 0$ and suppose that for $0 \leq j \leq n$ the functions $f_j$ satisfying (1) – (4) have been defined. By (2) there are $s_1, s_2, \cdots, s_N$ in $S_0$ such that the set $\bigcup_{k=1}^{N}\{w_k\}$ is a $2^{-n}$-net for $K_{n+1}$ where $w_k = f_n(s_k) \subset \text{int } K_n$ for $k = 1, 2, \cdots, N$. Let $Z = \bigcup_{j=1}^{M}\{z_j\}$ be a $3 \cdot 2^{-n-3}$-net for $K_{n+2}$ such that $|z_j| < 1 - 9 \cdot 2^{-n-4}$. We enumerate elements of $Z$ in such a way that

$$|w_k - z_j| < 2^{-n} \quad \text{for } M(k - 1) < j \leq M(k) \quad (k = 1, 2, \cdots, N)$$

where $0 = M(0) < M(1) < \cdots < M(N) = M$. Let

$$\sigma_1 = \max_{1 \leq k \leq N} \max_{M(k - 1) < j \leq M(k)} |w_k - z_j|; \quad \sigma_2 = \max_{1 \leq k \leq n} |w_k|,$$
(6) \( \epsilon = \min \left( 2^{-n} - \sigma_1; 1 - 2^{-n} - \sigma_2; 1 - \|f_n\| \right) \).

Then \( 2^{-n} > \epsilon > 0 \). Since \( f_n \) is uniformly continuous on \( S \) we can choose \( \delta > 0 \) such that for arbitrary \( s' \) and \( s'' \) in \( S \)

(7) if \( d(s', s'') < \delta \), then \( \left| f_n(s') - f_n(s'') \right| < \frac{1}{4} \epsilon \).

Since \( S_0 \) is a perfect set, there are \( t_j \) in \( S_0 \) such that \( d(s_k, t_j) < \frac{1}{2} \delta \) for \( M(k-1) < j \leq M(k) \); \( t_p \neq t_q \) for \( p \neq q \). Thus, since \( w_k = f_n(s_k) \), we get

(8) \( \left| f_n(t_j) - z_j \right| \leq \left| w_k - z_j \right| + \left| f_n(t_j) - f_n(s_k) \right| < 2^{-n} - \frac{3}{4} \epsilon \).

Let

\[ \delta_1 = \frac{1}{3} \min \left( \delta, \min \frac{d(t_p, t_0)}{p \neq q} \right) . \]

Choose \( g_j \) in \( A \) for \( j = 1, 2, \cdots, M \) such that

(9) \( \|g_j\| = |g_j(t_j)|; g_j(t_j) = z_j - f_n(t_j), \)

(10) if \( d(s, t_j) > \delta_1 \), then \( |g_j(s)| < (16M)^{-1} \epsilon \),

(11) if \( s \in F_m \), then \( |g_j(s)| < M^{-1} \left[ \frac{1}{2} (1 + \|f_m\|) - \|f_n(s)\| \right] \)

\((m = 1, 2, \cdots, n). \)

To construct \( g_j \), choose arbitrary \( h_j \) in \( A \) such that \( 1 = h_j(t_j) > |h_j(s)| \)
for \( s \in S \setminus \{t_j\} \). Since \( t_j \in B(A) \), such \( h_j \) exists \((j = 1, 2, \cdots, M) \). Then put

\[ g_j = [z_j - f_n(t_j)]^p \]

where the integer \( p(j) \) is chosen such that for \( j = 1, 2, \cdots, M \),

if \( d(s, t_j) > \delta_1 \), then \( |h_j(s)|^{p(j)} < (16M)^{-1} \epsilon \);

if \( s \in F_m \), then \( |h_j(s)|^{p(j)} < M^{-1} \inf_{s \in F_m} \left[ \frac{1}{2} (1 + \|f_m\|) - \|f_n(s)\| \right] \)

\((m = 1, 2, \cdots, n). \)

Let us set

(12) \[ g = \sum_{j=1}^{M} g_j; \quad f_{n+1} = f_n + g. \]

We shall estimate the norms of \( g \) and \( f_{n+1} \). Let \( s \in S \). Let us consider two cases:

1°. \( d(s, t_j) > \delta_1 \) for all \( j \). Then, by (6), (10) and (12),

\[ \left| g(s) \right| \leq \sum_{j=1}^{M} |g_j(s)| < \frac{\epsilon}{16} < 2^{-n} , \]

and

\[ \left| f_{n+1}(s) \right| \leq \left| f_n(s) \right| + \left| g(s) \right| \leq \|f_n\| + \epsilon/16 < 1. \]

2°. There is exactly the one index \( j(s) \) such that \( d(s, t_{j(s)}) \leq \delta_1 \). Then by (5), (6), (10) and (12)
\[ |g(s)| \leq \|g_j(s)\| + \sum_{j \neq j(s)} |g_j(s)| \leq |z_j(s) - f_n(t_j(s))| + \frac{\epsilon}{16} \]
\[ < 2^{-n} - \frac{\epsilon}{4}. \]

Let us choose \( k(s) \) such that \( M(k(s) - 1) < j(s) \leq M(k(s)) \). Then, since \( d(s_k(s), t_j(s)) \leq \epsilon \) and \( \delta_1 < \frac{\epsilon}{2} \), we get
\[ d(s_k(s), s) \leq d(s_k(s), t_j(s)) + d(s, t_j(s)) < \epsilon. \]

Thus \( |f_n(s_k(s)) - f_n(s)| < \frac{\epsilon}{4} \). Then by (7), (8) and (9),
\[ |f_{n+1}(s)| \leq |f_n(s)| + |g(s)| \leq |f_n(s_k(s))| + |f_n(s_k(s)) - f_n(s)| \]
\[ + |g(s)| < |w_k(s)| + 2^{-n}. \]

Thus since \( w_k(s) \subset \text{int} \ K_n \), \( |f_{n+1}(s)| < 1. \)

It follows from the choice of \( \delta_1 \) that, for every \( s \in S \), either 1° or 2° is satisfied. Thus
\[ \|f_{n+1}\| < 1 \quad \text{and} \quad \|f_{n+1} - f_n\| = \|g\| < 2^{-n}. \]

Now we shall check that the set \( \cup_{j=1}^M \{f_{n+1}(t_j)\} \subset \text{int} \ K_{n+1} \) is a \( 2^{-n-1} \)-net for \( K_{n+2} \). It follows from (9) that
\[ f_{n+1}(t_j) = (f_n(t_j) + g_j(t_j)) + \sum_{p \neq j} g_p(t_j) = z_j + \sum_{p \neq j} g_p(t_j). \]

Hence
\[ |f_{n+1}(t_j) - z_j| < 2^{-n-4} \quad (j = 1, 2, \cdots, M) \]
because by (10), \( \sum_{p \neq j} |g_p(t_j)| < 2^{-4} \epsilon < 2^{-n-4}. \)

Since \( |z_j| < 1 - 9 \cdot 2^{-n-4} \) for \( j = 1, 2, \cdots, M \) and since \( \cup_{j=1}^M \{z_j\} \) is a \( 3 \cdot 2^{-n-3} \)-net for \( K_{n+2} \), the inequality (13) implies that \( |f_{n+1}(t_j)| < 1 - 2^{-n-1} \) for \( j = 1, 2, \cdots, M \) and \( \cup_{j=1}^M \{f_{n+1}(t_j)\} \) is a \( 2^{-n-1} \)-net for \( K_{n+2} \).

Finally if \( s \in F_m \) \((m = 1, 2, \cdots, n)\), then by (11) \( \sum_{j=1}^M |g_j(s)| < \frac{1}{2} (1 + \|f_m\| - |f_m|). \) Therefore \( |f_{n+1}(s)| < \frac{1}{2} (1 + \|f_m\|). \) Clearly if \( s \in F_{n+1} \) then \( |f_{n+1}(s)| \leq \|f_{n+1}\| < \frac{1}{2} (1 + \|f_{n+1}\|), \) because we have already checked that \( \|f_{n+1}\| < 1. \) This completes the induction.

To complete the proof we define
\[ h = \lim_{n \to \infty} f_n. \]

It follows immediately from (1)–(4) that \( h \) has the desired properties.

**Corollary 1.** Let \( A \) be a function algebra on \( S \), \( S \) compact and
metric. Let $S_0$ be a proper perfect compact subset of $B(A)$. Then there is a function $g$ in $A$ which peaks on some uncountable subset of $S_0$.

**Proof.** Let $\Delta$ be a closed uncountable subset of the unit circle of (1-dimensional) Lebesgue measure zero. Let $\phi$ be a continuous function on $K$ analytic at each point of int $K$ and such that $\phi^{-1}(1) = \Delta$ and $\phi^{-1}(K \setminus \Delta) \subset \text{int } K$ (cf. [9, p. 81]). Let $g = \phi \circ h$, where $h \in A$ is chosen as in Proposition 1. Since $\phi$ is a uniform limit (in $K$) of a sequence of polynomials, say $(W_n)$, the function $g$ is the uniform limit of the sequence $(W_n \circ h)$. Hence $g$ belongs to $A$.

Clearly $g$ peaks on the set $h^{-1}(\Delta) \subset S_0$.

4. The main result.

**Theorem 1.** Let $A$ be a function algebra on a compact metric space $S$. Then the following conditions are equivalent.

(a) $S$ is uncountable.
(b) $A^*$ is nonseparable.
(c) $B(A)$ is uncountable.
(d) There exists a subset $T$ of $S$ homeomorphic to the Cantor discontinuum and a linear operator $u : C(T) \to A$ such that for every $f \in C(T)$, $\|uf\| = \|f\|$ and $(uf)(s) = f(s)$ for $s \in T$.
(e) $A$ contains a closed linear subspace which is isometrically isomorphic to the space of all continuous functions on the Cantor discontinuum and which is the range of a projection of norm one from $A$.
(f) Every separable normed linear space is linearly isometric to a linear subspace of $A$.

**Proof.** (a)$\Rightarrow$ (b). If $S$ is uncountable, then $M(S) = [C(S)]^*$ is nonseparable (because if $s' \neq s''$, then $\|\eta_{s'} - \eta_{s''}\| = 2$ for arbitrary $s'$ and $s''$ in $S$). Therefore we can restrict our attention to the case where $A \neq C(S)$. In this case there is a maximal antisymmetric set, say $S_1$, for $A$ which contains more than one point (cf. [2], [6]). Let $r$ be the operator which assigns to each $f$ in $A$ its restriction to $S_1$ and let $rA = A_1$. By results of Bishop [2] (cf. also [6]) and Šilov [17], $A_1$ is an antisymmetric function algebra on $S_1$. Since $r$ maps $A$ onto $A_1$, the adjoint map $r^*$ is a linear homeomorphism from $A_1^*$ into $A^*$ [5, p. 488]. Therefore it is enough to show that $A_1^*$ is nonseparable. To prove this observe first that $B(A_1)$ has no isolated points. Otherwise, $A_1$ would contain a nonconstant idempotent which contradicts the antisymmetry of $A_1$. (Indeed, let $X$ be a function algebra on $T$ and let $t_0$ be an isolated point of $B(X)$. Then [15, p. 53] there is an $f$ in $X$ such that $\|f\| = f(t_0) = 1$ and $|f(t)| \leq \frac{3}{2}$ for $t \neq t_0$. Clearly $\|f^p - f^q\| \leq (\frac{3}{2})^p + (\frac{3}{2})^q$ for $p, q = 1, 2, \ldots$. Hence $(f^n)$ is a Cauchy sequence. Let $e = \lim_n f^n$; then
Thus $B(A_1)$ is uncountable, because it has no isolated points and it is a $G_δ$ in a compact metric space (cf. [8, p. 137, X]).

Let us set

$$x_s^* = f(s) \quad \text{for } f \in A_1 \text{ and } s \in B(A_1).$$

Then $x_s^* \in A_1^*$ and $\|x_s^*\| \leq 1$ for $s \in B(A_1)$, and $\|x_s^* - x_t^*\| = 2$ for $s_1 \neq s_2$. Indeed, if $f_i$ peaks at $s_i$ ($i = 1, 2$), then

$$\lim_{n} \|f_1^n - f_2^n\| = 1 \quad \text{and} \quad 2 \geq \|x_{s_1}^* - x_{s_2}^*\| \geq \lim_{n} (x_{s_1}^* - x_{s_2}^*)(f_1^n - f_2^n) = 2.$$

Thus $A^*$ is nonseparable.

non (c) $\implies$ non (b). Let us set

$$M(B(A)) = \left\{ \mu \in M(S) : \mu(f) = \int_{B(A)} f d\mu \right\}.$$  

Since (by non (c)) $B(A)$ is countable, $M(B(A))$ is isometrically isomorphic to the space $l_1$ of all absolutely convergent series. Hence $M(B(A))$ is separable. For $\mu \in M(B(A))$, let $\mu_A$ denote the restriction of the linear functional $\mu$ to $A$. Then, by the Bishop-Choquet-de Leeuw set up theorem (cf. [15, p. 38 and p. 53]) the correspondence $\mu \rightarrow \mu_A$ is a linear map from $M(B(A))$ onto $A^*$. Hence, $A^*$ is separable as a continuous image of a separable Banach space.

(c) $\implies$ (d). According to a result of Bishop [3] and the Main Theorem of [12] it is enough to show

(d') there is an uncountable compact subset $T$ of $S$ such that if $\mu \in A^+ = \{ \nu \in M(S) : \nu(f) = 0 \text{ for } f \in A \}$, then $\mu(E) = 0$ for arbitrary Borel set $E \subset T$.

Since $B(A)$ is an uncountable absolute $G_δ$, it contains a proper compact perfect subset $S_0$ (cf. [8, p. 138, XI]). Choose (using Corollary 1) $g = g_{0,0}$ in $A$ such that $\|g_{0,0}\| = 1$ and $g$ peaks on an uncountable subset of $S_0$. Let us set $S_{0,0} = g_{0,0}^{-1}(1)$. Since $S_{0,0}$ is uncountable we can find disjoint perfect subsets of $S_{0,0}$, say $S_{1,0}$ and $S_{1,1}$ such that $\text{diam } S_{i,0} < \frac{1}{2}$ for $i = 0, 1$. Thus, applying Corollary 1, we can find functions $g_{1,0}$ and $g_{1,1}$ in $A$ which have norm one and which peak on some uncountable subsets of $S_{1,0}$ and $S_{1,1}$ respectively. We put $S_{1,0} = g_{1,0}^{-1}(1)$ for $i = 0, 1$. Continuing in this manner we define inductively for $j = 0, 1, \ldots , 2^n - 1$ and for $n = 0, 1, \ldots$ uncountable closed subsets $S_{n,j}$ of $C(A)$ and functions $g_{n,j}$ in $A$ with the following properties:

(i) $\|g_{n,j}\| = 1$ and $g_{n,j}$ peaks on $S_{n,j}$.

4 If $T$ is a subset of a metric space $S$, then $\text{diam } T = \sup_{s, t \in T} d(s, t)$.
(ii) \( S_{n,j} \cap S_{n,k} = \emptyset \) for \( k \neq j \),
(iii) \( S_{n+1,2j} \cup S_{n+1,2j+1} \subseteq S_{n,j} \),
(iv) \( \text{diam } S_{n,j} \leq (n+1)^{-1} \) for \( n > 0 \).

Let us set
\[
T = \bigcap_{n=0}^{\infty} \bigcup_{j=0}^{2^n-1} S_{n,j}.
\]

Clearly \( T \) is homeomorphic to the Cantor discontinuum (as a proper zero-dimensional perfect compact set [10, Vol. II, p. 58]). We shall show that \( T \) satisfies the condition \( (d') \). To do this it is enough to show that if \( \mu \in A^{\perp} \), then \( \mu(T \cap S_{m,k}) = 0 \) for \( k = 0, 1, \ldots, 2^{m-1} \) and for \( m = 0, 1, \ldots \), because the sets \( S_{m,k} \) generate the field of all Borel subsets of \( T \).

For a fixed pair \( (m, k) \) with \( 0 \leq k \leq 2^m \) and \( m = 0, 1, \ldots \), let us set
\[
N_p = \{ j = 2^p - 1 : S_{p,j} \subseteq S_{m,k} \} \quad (p = m, m + 1, \ldots).
\]
Then
\[
T \cap S_{m,k} = \bigcap_{n \geq m} \bigcup_{j \in N_p} S_{p,j}.
\]

Let \( \varepsilon > 0 \). Since \( \mu \) is a finite regular Borel measure, there is an index \( p \geq m \) such that
\[
\mu \left( \bigcup_{j \in N_p} S_{p,j} \setminus T \cap S_{m,k} \right) < \varepsilon.
\]

Let us set
\[
f_n = \sum_{j \in N_p} g_{p,j}^n \quad (n = 1, 2, \ldots).
\]
Clearly, by (i), \( \|f_n\| \leq \sum_{j=0}^{2^n-1} \|g_{p,j}^n\| = 2^p - 1 \) for \( n = 1, 2, \ldots \) and \( g_{p,j}^n \) converges pointwise to the characteristic function of \( S_{p,j} \) \( (j = 0, 1, \ldots, 2^p - 1) \). Hence, by (ii), the sequence \( f_n \) converges to the characteristic function of the set \( \bigcup_{j \in N_p} S_{p,j} \). Therefore, since \( \mu \in A^{\perp} \), the Lebesgue dominated convergence theorem implies
\[
0 = \lim_n \mu(f_n) = \lim_n \int_S f_n(t) \mu(dt) = \int_S \lim_n f_n(t) \mu(dt) = \mu \left( \bigcup_{j \in N_p} S_{p,j} \right).
\]
Thus \( |\mu(T \cap S_{m,k})| < \varepsilon \), because of the choice of the index \( p \). Since \( \varepsilon \) is an arbitrary positive number, \( \mu(T \cap S_{m,k}) = 0 \).

(d) \( \rightarrow \) (e). This follows from [12, Proposition 1].

(e) \( \rightarrow \) (f). This follows from the fact that every separable normed
linear space is linearly isometric to a linear subspace of the space of all continuous functions on the Cantor discontinuum [4, p. 93 (6)].

non (a) → non (f). If \( S \) is countable, then \( M(S) \) is linearly isometric to \( l_1 \), in particular \( M(S) \) is separable [16], [17]. Thus every linear subspace \( X \) of \( C(S) \), in particular every subspace of \( A \), has a separable dual, because \( X^* \) is a continuous image (by the restriction map) of \( M(S) \).

**Corollary 2.** Let \( A \) be a function algebra on a compact metric space \( S \) and suppose that there exists a (bounded) projection from \( C(S) \) onto \( A \). Then \( A \) is linearly homeomorphic to \( C(S) \).

**Proof.** If \( S \) is countable, then by a result of Rudin [16], \( A = C(S) \). If \( S \) is uncountable, then by a result of Miljutin (cf. [11], [13, Theorem 8.5]) \( C(S) \) is linearly homeomorphic to \( C(\mathcal{C}) \), where \( \mathcal{C} \) denotes the Cantor discontinuum. Thus by the assumption of the corollary, \( A \) is linearly homeomorphic to a complemented subspace of \( C(\mathcal{C}) \). On the other hand, by Theorem 1 (condition (e)) \( A \) contains a complemented subspace which is linearly isometric to \( C(\mathcal{C}) \). Thus, by [13, Proposition 8.3], \( A \) is linearly homeomorphic to \( C(\mathcal{C}) \) or equivalently to \( C(S) \).

Let us note that Corollary 2 is related to the conjecture of Glicksberg [7], that if \( A \) is a closed complemented subalgebra of \( C(S) \), \( S \) compact Hausdorff, then \( A \) is selfadjoint. Indeed if this conjecture is true, then the Stone-Weierstrass theorem will imply that a complemented function algebra on a compact Hausdorff space \( S \) is the whole space \( C(S) \).

**References**


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* A linear subspace \( Y \) of a Banach space \( X \) is said to be complemented in \( X \) if there is a (bounded linear) projection from \( X \) onto \( Y \).


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