ABSOLUTE BAIRE SETS

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1. Introduction. Definitions and facts. For background in notation and terminology the reader should consult [2]. We make here the blanket assumption that all spaces to be considered in this note are completely regular Hausdorff topological spaces.

A continuous map $f: X \to Y$ is called proper if f is onto, closed, continuous, and if for every compact set K in Y, $f^{-1}(K)$ is compact.

The family of Baire sets of a space X is the σ-field generated by the zero-sets of X. The family of Borel sets of a space X is the σ-field generated by the closed sets of X. For a metric space the two families coincide, but in general the family of Borel sets properly contains the family of Baire sets.

A subset A of a space X is said to be C^*-embedded in X if every real-valued continuous bounded function on A is the restriction of a real-valued continuous (bounded) function on X. Thus, for example, a space X is C^*-embedded in its Stone-Čech compactification βX.

2. Absolute Baire spaces. We will now prove our first main result, which characterizes the spaces which are Baire sets in their Stone-Čech compactifications. The characterization, which uses the idea included in the proof of Theorem D, §51, of Halmos's text on measure theory [2], is surprisingly simple.

2.1. THEOREM. *The following are equivalent on any space* X:

(1) *X is Baire in βX (i.e. X is an "absolute Baire set").*

(2) *There is a separable metric space Γ which is an absolute Borel space (in the class of separable metric spaces) and a proper mapping $\phi: X \to \Gamma$.*

PROOF. (1)$\Rightarrow$(2). Suppose that X is a Baire set in βX. There is a sequence of zero-sets of βX, say Z_n, $n=1, 2, \cdots$, such that X belongs to the σ-field generated by $\{Z_n, n=1, 2, \cdots\}$. We choose $f_n \in C(\beta X)$ such that $0 \leqq f_n \leqq 1$ and $Z_n = Z(f_n)$, and we define $d(x, y) = \sum_{n=1}^{\infty} (1/2^n) |f_n(x) - f_n(y)|$. Clearly d is a continuous pseudometric on βX. Let Ξ be the metric space associated with the pseudometric space $(\beta X, d)$. The identity mapping $\pi: \beta X \to \Xi$ is continuous and onto, and, therefore, Ξ is a compact (metric) space. Arguing as in Halmos, we conclude that there is a subset Γ_n of Ξ such that $\pi^{-1}\Gamma_n$

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$=Z_n$ for $n=1, 2, \cdots$. This clearly implies that there is some Γ in the σ-field generated by $\{\Gamma_n, n=1, 2, \cdots\}$ such that $\pi^{-1}\Gamma=X$. Now $\pi\pi^{-1}\Gamma_n=\Gamma_n=\pi Z_n$; hence, Γ_n is compact, and thus a Borel set of Ξ; hence Γ is also a Borel set of Ξ. Thus, Γ is a separable metric space which is an absolute Borel space in the class of separable metric spaces. It is easily seen that the mapping $\pi: X \rightarrow \Gamma$ is proper.

(2)$\Rightarrow$(1). Suppose that there is a proper mapping $\pi: X \rightarrow \Gamma$, where Γ is an absolute Borel separable metric space. Let Δ be a compact metric space which is a compactification of Γ. (Such a space Δ exists; e.g. we embed Γ in a countable product of unit intervals and let Δ be its closure.) Let $\bar{\pi}: \beta X \rightarrow \Delta$ be the Stone-Čech extension of π. By [4, Lemma 1.5], $\bar{\pi}(\beta X - X)=\Delta-\Gamma$ and, hence, $\bar{\pi}^{-1}(\Gamma)=X$. Thus, X is a Baire set in βX. (This is almost immediate because the inverse image under a continuous mapping of a zero set is again a zero set; and Γ, being an absolute Borel set, is a Borel subset of Δ.)

2.2. Remark. The author has not been able to prove thus far that the following conditions (3), (4) are also equivalent to (1):

(3) X is a Baire set in some compactification of X,

(4) X is a Baire set in every compactification of X. Of course, it is true that (4)$\Rightarrow$(3)$\Rightarrow$(1).

2.3. Remark. We could also define Baire sets of class α and relate them to the separable metric absolute Borel spaces of class α; such a classification will not be undertaken here.

2.4. Corollary. *If X is a Baire set in βX, then X is a Lindelöf space.*

Proof. Use Theorem 2.1 and [4, Theorem 2.2].

2.5. Corollary. *A metric space which is a Baire set in βX is separable.*

2.6. Theorem. *Let A be a Baire set in a compact space. Then A is a Baire set in βX; in particular, every Baire set in a compact space is Lindelöf.*

Proof. The proof is similar to that of the first half of Theorem 2.1, and it is omitted.

The proof of the following lemma is omitted.

2.7. Lemma. *Let $A \subset B \subset C$ be inclusions of spaces, with B C^*-embedded in C. If A is a Baire set in B, and B is a Baire set in C, then A is a Baire set in C.*

2.8. THEOREM. *Let X be an absolute Baire set. Then every Baire set of X is an absolute Baire set.*

PROOF. A Baire set of X is, by 2.7, a Baire set of βX. Hence, by 2.6, it is an absolute Baire set.

2.9. LEMMA. *Let X_n be a sequence of absolute Baire sets. Then the product $X=\Pi_n X_n$ is also an absolute Baire set.*

PROOF. We know that X_n is a Baire set in βX_n. Let $BX=\Pi_n \beta X_n$; clearly, $X=\bigcap_n \Pi_n^{-1}(X_n)$, and, thus, X is easily seen to be a Baire set in βX; hence, X is a Baire set in βX.

2.10. THEOREM. *A countable product of absolute Baire spaces is an absolute Baire space, and, thus, Lindelöf.*

2.11. COROLLARY. *The product of a sequence of locally compact, σ-compact spaces is Lindelöf.*

2.12. COROLLARY. *If X is an absolute Baire space, then $\beta X-X$ is Lindelöf.*

2.13. EXAMPLE. A σ-compact space which is not an absolute Baire space.

Let N be the set of positive integers, and let $p\in\beta N-N$. The set $\Sigma=N\cup(p)$ is countable and, hence, σ-compact; in fact, every point is a G_δ. But $\beta\Sigma-\Sigma=\beta N-N-(p)$ is easily seen to be a pseudocompact space, which is not compact, and, hence, certainly not a Baire set in βN. By 2.12, Σ is not an absolute Baire space.

3. Baire sets in $\beta X-X$.

3.1. The main objective in this paragraph is to show that if X is locally compact, σ-compact, then every Baire set of $\beta X-X$ is C^*-embedded. There is some background to this theorem. Gillman and Henriksen [1] have proved that if X is as above, then every complement of a zero-set in $\beta X-X$ is C^*-embedded. Since a complement of a zero-set is certainly a Baire set, the following theorem includes this result.

3.2. THEOREM. *If X is a locally compact, σ-compact space, then every Baire set of $\beta X-X$ is C^*-embedded in βX.*

PROOF. Let A be a Baire set in $\beta X-X$. Notice that the fact that X is a locally compact, σ-compact space implies that $\beta X-X$ is a zero-set in βX, and, hence, a C^*-embedded Baire set in βX. Therefore, by 2.7, A is a Baire set in βX. Hence, the set $A \cup X$ is also a Baire set in βX; thus, by 2.6, $A \cup X$ is a Lindelöf space, and in particular, a normal space. Notice further that A is closed in the relative topology of $A \cup X$, and, hence, it is C^*-embedded in $A \cup X$. Further, since $X \subset A \cup X \subset \beta X$, we see that $A \cup X$ is C^*-embedded in βX. By transitivity, A is C^*-embedded in βX, and a fortiori, in $\beta X - X$.

3.3. REMARK. It is instructive to see how we can prove the Gillman-Henriksen theorem mentioned above by a direct argument. In fact, let A be a cozero-set of $\beta X - X$; since $\beta X - X$ is compact, A is σ-compact, and, hence, $A \cup X$ is also σ-compact, and, therefore, normal. The rest of the argument proceeds as above.[1]

REFERENCES

1. L. Gillman and M. Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. **82** (1956), 366–391.

2. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.

3. P. R. Halmos, *Measure theory*, Van Nostrand, New York, 1950.

4. M. Henriksen and J. R. Isbell, *Some properties of compactifications*, Duke Math. J. **25** (1958), 83–105.

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[1] NOTE: It has come to the attention of the author, after the completion of this paper, that Z. Frolik, in his paper *On coanalytic and bianalytic spaces*, Bull. Acad. Polon. Sci. **12** (1964), 527–530, has obtained a different proof of Theorem 2.1.