A NOTE ON CERTAIN GENERALIZED FREE PRODUCTS

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Let $A\pi B|U$ be the generalized free product of A and B with amalgamated subgroup U [2]. Recall that a group G is residually P for some group property P, if, for any element $g\in G$ not the identity, there exists a normal subgroup N not containing g such that G/N has property P.

Baumslag [1] has shown that if A and B are finitely generated torsion free nilpotent groups and U is cyclic, then $A\pi B|U$ is residually finite. However, it can be verified that $G=gp(a, b|a^2=b^3)$ is a generalized free product of the above type which is not residually a finite p-group for any prime p. Hence we might ask what possible finite indices normal subgroups in $A\pi B|U$ might have.

The following theorem, which grew out of a conversation with Thomas Head, shows that normal subgroups of all possible finite indices exist in $A\pi B|U$, when U is cyclic.

THEOREM. *Let A and B be finitely generated nontrivial torsion-free nilpotent groups and let U be cyclic. Then $S=A\pi B|U$ has an infinite cyclic factor group.*

PROOF. *Case* 1. Suppose A and B are cyclic, say $S=gp(a, b|a^h=b^k)$. Then the abelianization of S, S/S', is an infinite abelian group and so S/S' has an infinite cyclic factor group. Hence so does S.

Case 2. Let A and B both be abelian groups, but not both cyclic. Let H and K be the isolators of U in A and B, respectively. The groups H and K are direct factors of A and B, respectively (see [2], isolated subgroups), say $A=C\times H$, $B=D\times K$. Either $C\neq 1$ or $D\neq 1$, say $C\neq 1$. The natural epimorphisms of A onto C and B onto 1 can be extended to an epimorphism of S onto C, because the amalgamated subgroup U maps onto the trivial group in C. C is a finitely generated torsion-free abelian group, so C has an infinite cyclic factor group, and so must S.

Case 3. Now suppose the nilpotent classes of A and B are n and m (>0), respectively. Let $U=\langle u\rangle$ and suppose either $u\in Z_{n-1}(A)$ or $u\in Z_{m-1}(B)$, where $Z_{n-1}(A)$ is the last proper term of the upper central series of A. Say $u\in Z_{n-1}(A)$. In this case we can extend the natural epimorphisms $A\rightarrow A/Z_{n-1}(A)$ and $B\rightarrow 1$ to an epimorphism of S onto the torsion-free abelian group $A/Z_{n-1}(A)$, for the amalgamated subgroup U maps onto the identity.

Received by the editors July 19, 1966.

Finally, suppose $u \notin Z_{n-1}(A)$ and $u \notin Z_{m-1}(B)$. Since $A/Z_{n-1}(A)$ and $B/Z_{m-1}(B)$ are torsion free $U \cap Z_{n-1}(A) = 1 = U \cap Z_{m-1}(B)$. Hence there is an epimorphism of S onto $G = A/Z_{n-1}(A)\pi B/Z_{m-1}(B) \mid U$, and G, by either Case 1 or Case 2, has an infinite cyclic factor group. This completes the proof.

COROLLARY. *Let A and B be groups with normal subgroups N and M, respectively, such that A/N and B/M are finitely generated nontrivial torsion-free nilpotent groups. Then $S = A\pi B \mid U$, U cyclic, has an infinite cyclic factor group.*

PROOF. Let $U = (u)$, and suppose either $u \in N$ or $u \in M$. Say $u \in N$. Then we can extend the epimorphisms $A \to A/N$ and $B \to 1$, to an epimorphism of S onto A/N. Hence S has an infinite cyclic factor group.

If $u \notin N$ and $u \notin M$, then since A/N and B/M are torsion free, $U \cap N = 1 = U \cap M$. Hence we can find a natural epimorphism of S onto $A/N\pi B/M \mid U$, and the result follows by the theorem.

This corollary says that if A and B have infinite cyclic factor groups, so does $A\pi B \mid U$. This probably is the main result here, but our proofs of the cases considered suggest possible noncyclic homomorphic images.

We mention a final special case.

COROLLARY. *Let A and B be free groups. If U is cyclic, then $S = A\pi B \mid U$ has an infinite cyclic factor group.*

REFERENCES

1. G. Baumslag, *On the residual finiteness of generalized free products of nilpotent groups*, Trans. Amer. Math. Soc. **106** (1963), 193–209.

2. A. Kurosh, *The theory of groups*, Vol. II, Chelsea, New York, 1960.

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