HOMOMORPHIC RETRACTIONS IN SEMIGROUPS

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1. **Introduction.** It has been shown by Wallace [7], [8], that the minimal ideal of a compact semigroup is a retract of the semigroup. The purpose of this paper is to determine when the minimal ideal of a semigroup is a homomorphic retract of the semigroup, i.e., there exists a retraction which is also a homomorphism of the semigroup onto its minimal ideal. This turns out to be an algebraic problem since it will be shown that any homomorphic extension of the identity function of the minimal ideal to the semigroup is continuous, providing that the minimal ideal is completely simple and each maximal subgroup of the minimal ideal is a topological group. To emphasize these facts, semigroups with the discrete topology will be used in the main theorem which gives some conditions which are necessary and sufficient for a completely simple minimal ideal of a discrete semigroup to be a homomorphic retract.

The various structure theorems for a completely simple minimal ideal which are used here may be found in [2], [4], [5], [6], [7] and [8].

A *homomorphic retraction* is a function from S into S which is both a retraction and a homomorphism. The image of the homomorphic retraction is called a *homomorphic retract.* These terms seem to have first appeared in [1] and have been investigated in [3]. The concept of homomorphic retraction appears in [4], where the question of the extendability of homomorphisms is investigated. The following proposition shows that the question of extending homomorphisms is closely related to the concept of homomorphic retractions.

PROPOSITION 1. *If A is a subsemigroup of S, then A is a homomorphic retract of S if and only if each continuous homomorphism defined on A can be extended to all of S.*

This proposition follows easily from the definition of homomorphic retraction.

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2. **The main theorem.** Let S be a semigroup with a completely simple minimal ideal K. Let $e_0 \in E \cap K$ (E is the set of idempotents of

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S) and let $Z(e_0)=(Se_0\cap E)\times e_0Se_0\times(e_0S\cap E)$ be the semigroup with the multiplication $(e, g, f)(e', g', f')=(e, gfe'g, f')$. The Rees-Suschkewitsch Theorem [6] states that the function ψ: $Z(e_0)\rightarrow K$ defined by $\psi(e, g, f)=egf$ is an isomorphism.

Let u: $K\rightarrow K\cap E$ be defined by $u(x)$ is the unique identity of x [8]. Let v: $K\rightarrow K$ be defined by $v(x)$ is the inverse of x in the group $u(x)Su(x)$ [8]. Define ϕ: $S\rightarrow Z(e_0)$ by $\phi(x)=(xv(e_0xe_0), e_0xe_0, v(e_0xe_0)x)$. It is a routine computation to verify that the range of ϕ is $Z(e_0)$. Let π: $Z(e_0)\rightarrow(Se_0\cap E)\times(e_0S\cap E)$ be defined by $\pi(e, g, f)=(e, f)$. It is clear that π is a homomorphism.

THEOREM 2. *If S is a semigroup with a completely simple minimal ideal, then the following diagram is commutative:*

$$\begin{array}{ccc} K & \subset & S \\ \cap & & \downarrow\phi \\ K & \overset{\psi}{\leftarrow} & Z(e_0) \end{array}$$

Moreover, if S is discrete, then the following statements are equivalent:

(1) *ϕ is a homomorphism.*

(2) *K is a homomorphic retract of S.*

(3) *There is a function f: $S\rightarrow K\cap E$ such that $e=f(x)$ iff $ex=ex$ and $f(xy)xyf(xy)=f(x)xyf(y)$ for all $x, y\in S$ and $e\in K\cap E$.*

(4) *There exist functions g, g': $S\rightarrow K$ such that $g(x)S=xK$ and $Sg'(x)=Kx$.*

(5) *Let $e, f\in K\cap E$. If $ef=e$ then $u(xe)=u(xf)$ and if $ef=f$, then $u(ex)=u(fx)$ for all $x\in S$* [3].

(6) *There exists a retraction f: $S\rightarrow K\cap E$ such that $f(xy)=f(f(x)f(y))$ for all $x, y\in S$.*

(7) *S is a rectangular band of homogroups whose minimal ideals are the maximal subgroups of K* [3].

PROOF. Since $\psi^{-1}(x)=(u(xe_0), e_0xe_0, u(e_0x))$ [8], $\phi|K=\psi^{-1}$ and the diagram is commutative.

(1) implies (2) is immediate from Proposition 1.

(2) implies (1). Let r: $S\rightarrow K$ be a homomorphic retraction. Then $\psi^{-1}r$ is a homomorphism of S onto $Z(e_0)$, but $\psi^{-1}r=\phi$.

(2) implies (3). Let r: $s\rightarrow K$ be a homomorphic retraction and define $f(x)=u(r(x))$. Thus $f(x)x=r(x)=xf(x)$. If $e\in K\cap E$, $x\in S$, and $xe=ex$, then $ef(x)=(ex)(vr(x))$ and $f(x)e=(vr(x))xe$ so that $eSe=f(x)Sf(x)$. If $x, y\in S$, then $f(xy)xyf(xy)=r(xy)=f(x)xyf(y)$.

(3) implies (4). Let $f=g=g'$. Thus, if $x\in S$ and $k\in K$, then $xk=f(xk)xk=f(x)xkf(k)\in f(x)S$ so that $xK=g(x)S$. Similarly, $Sg'(x)=Kx$.

(4) implies (5). If $e, f \in K \cap E$ and $ef = e$, then $Sf = Se$ and $xeS = g(x)S = xfS$ so that $u(xe) = u(xf)$ for all $x \in S$. The other statement is the dual.

(5) implies (6). Define $f: S \to K \cap E$ by $f(x) = u(xe_0x)$. It is clear that f is a retraction. It is easily verified that $xkS = xe_0S$ and $Skx = Se_0x$ for all $x \in S$ and $k \in K$. Thus $u(xe_0x)u(ye_0y)e_0u(xe_0x)u(ye_0y) \in xe_0S \cap Se_0y$ so that $f(xy) = f(f(x)f(y))$.

(6) implies (7). It is clear that $\pi\psi^{-1}f$ is a homomorphism of S onto the rectangular semigroup $(Se_0 \cap E) \times (e_0S \cap E)$. Since for all $(x, y) \in (Se_0 \cap E) \times (e_0S \cap E)$, $(\pi\psi^{-1}f)^{-1}(x, y) = f^{-1}u(xy)$, $\{(\pi\psi^{-1}f)^{-1}(x, y) \mid (x, y) \in (Se_0 \cap E) \times (e_0S \cap E)\}$ is a partition of S into homogroups whose minimal ideals are the maximal subgroups of K.

(7) implies (2) is in [3].

COROLLARY 3. *If S is a semigroup with a completely simple minimal ideal K such that it is continuous, then statements* (1)–(7) *of Theorem* 2 *are equivalent. Thus, if S is a compact semigroup, then statements* (1)–(7) *of Theorem* 2 *are equivalent.*

This follows from the fact that in this case ϕ is continuous.

COROLLARY 4. *If S has a unit, then K is a homomorphic retract of S if and only if K is a group.*

This is immediate from (3) of Theorem 2.

The following proposition gives the homomorphic retraction properties of the factors of $Z(e_0)$ in the Rees-Suschkewitsch Theorem.

PROPOSITION 5.

(a) *$Se_0 \cap E$ is a homomorphic retract of K.*

(b) *$e_0S \cap E$ is a homomorphic retract of K.*

(c) *e_0Se_0 is a homomorphic retract of S if and only if the multiplication in $Z(e_0)$ agrees with the coordinatewise multiplication, i.e., if $p \in (Se \cap E)$ and $q \in (eS \cap E)$, then $qp = e$.*

It should be noted that the multiplication in $Z(e_0)$ is co-ordinatewise whenever K is a minimal left ideal or minimal right ideal.

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AN EXAMPLE OF TWO UNIFORMITIES EQUAL IN HEIGHT AND PROXIMITY

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1. Introduction. In [7] Yu. M. Smirnov first raised the question whether there existed a proximity class without a largest member. The first example was given in [2] and later examples occurred in [4] and [6].

In [5] the author introduced the concept of height of uniformities and showed that if two uniformities in the same proximity class were comparable in height but not comparable in the usual ordering, then their least upper bound was not in that proximity class. An example was given. This example was new in the sense that all previous examples had involved pairs of uniformities which were not comparable in height.

The question was also raised whether two uniformities exist which are equal in both proximity and height. This question is answered by the example given in this paper.

In §2 we review what we need of proximity and height. In §3 we construct the subbasic covers which are used in §4 to complete the construction of the two uniformities.

2. Proximity and height. The concept of proximity was first introduced by Efremovič in [3] and more about it occurs in [1], [2], and [6]. We say a uniformity $\mathcal{U}$ is $\leq^p$ to a uniformity $\mathcal{V}$ if, for any set A in X and any U in $\mathcal{U}$, there exists V in $\mathcal{V}$ such that $\mathrm{St}(A, V) \subset \mathrm{St}(A, U)$, where $\mathrm{St}(A, U)$ stands for star of A with respect to U. If both

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