PURELY INDECOMPOSABLE TORSION-FREE GROUPS

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1. Introduction. We consider the following question posed by E. Weinberg [8]: Does there exist a torsion-free abelian group of power greater than the continuum ($\aleph$) with the property that each pure subgroup is indecomposable? J. W. Armstrong [1] has given a negative answer to this question for a class of groups somewhat larger than the class of homogeneous groups. In §2 we show that in general the answer to Weinberg's question is negative. Furthermore we show that every reduced torsion-free purely indecomposable group is isomorphic to a subgroup of $J = \prod_p I_p$ where p ranges over the primes and I_p denotes the p-adic group. In §3 we characterize the pure subgroups of J which are purely indecomposable and in §4 we show there are exactly $2^{\aleph}$ nonisomorphic torsion-free purely indecomposable groups.

Throughout this note all groups are abelian and we follow [4] in notation. The notion of a cotorsion group introduced by Harrison in [7] plays an important role in this paper. Some basic properties of cotorsion groups are listed in [5].

Definition. A purely indecomposable group is a group in which every pure subgroup is indecomposable.

2. The power of a torsion-free purely indecomposable group.

Lemma. *Let E be a torsion-free cotorsion group and let x be an element of E. Then there is a direct summand of E containing x of power not exceeding $\aleph$.*

Proof. It suffices to consider $x \neq 0$. Let $A = \{x\}_*$, the pure closure in E of the cyclic group $\{x\}$. Then A is of rank one, which implies $|A| = \aleph_0$. Now E/A is torsion-free since A is a pure subgroup of E. Therefore, $E/A = H/A + K/A$ where K/A is reduced and torsion-free and H/A is torsion-free divisible. It follows that

$$E/H = E/A/H/A = K/A$$

is a reduced torsion-free group. Since E/H is reduced and E is cotorsion, it is well known that H must also be cotorsion. Thus Ext $(E/H, H) = 0$ and hence H is a direct summand of E. Since H/A is divisible, A is

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dense in the n-adic topology on H. However, the n-adic topology on a reduced torsion-free group is Hausdorff; thus it follows that $|H| \leqq |A|^{\aleph_0} = (\aleph_0)^{\aleph_0} = \aleph$.

THEOREM. *If G is a torsion-free purely indecomposable group, then the power of G does not exceed $\aleph$.*

PROOF. If G has a nontrivial divisible subgroup, it is immediate that G must be isomorphic to the additive group of rational numbers if it is purely indecomposable.

Thus we may assume that G is reduced and also that $G \neq 0$. Embed G as a pure subgroup of a torsion-free cotorsion group E. Let $x \neq 0$ be an element of G. By the above lemma, there is a direct summand H of E such that $x \in H$ and $|H| \leqq \aleph$. Let $E = M + H$ and $S = M \cap G + H \cap G$. Since M, H and G are pure subgroups of E and since E is torsion-free, it follows that $M \cap G$ is pure in M and $H \cap G$ is pure in H. Thus, $E = M + H$ implies that S is a pure subgroup of E; hence S is a pure subgroup of G. Therefore, since G is purely indecomposable, we have that $M \cap G = 0$ or $H \cap G = 0$. But $x \neq 0$ in $H \cap G$ implies that $M \cap G = 0$. Let π be the natural projection of $E = M + H$ onto H. Since $M \cap G = 0$, π restricted to G is a monomorphism. Thus $|G| = |\pi(G)| \leqq |H| \leqq \aleph$.

As in the introduction, let $J = \prod_p I_p$ where p ranges over the primes and I_p denotes the p-adic group. We have the following.

COROLLARY. *If G is a reduced, torsion-free, purely indecomposable group, then G can be embedded in the n-adic completion of a reduced torsion-free group of rank one. Thus G is isomorphic to a subgroup of J.*

PROOF. We use the notation in the proof of the above theorem. Since H is a reduced torsion-free cotorsion group, it follows that H is algebraically compact and thus complete in its n-adic topology. Since $H/\{x\}_*$ is divisible, $\{x\}_*$ is of rank one, and $\pi: G \rightarrow H$ is a monomorphism, the first part of the corollary is immediate. It is also known that $H = \prod_p A_p$ where, for each prime p, A_p is a p-adic module complete in its p-adic topology. Each p-basic subgroup of H is cyclic since $H/\{x\}_*$ is divisible and each p-basic subgroup of $\{x\}_*$ is cyclic. (For the notion of a p-basic subgroup, see [6].) Thus A_p is indecomposable as a p-adic module. It follows that $A_p = 0$ or $A_p = I_p$ and that H is isomorphic to a direct summand of J, which completes the proof.

3. Pure subgroups of J which are purely indecomposable.

We now consider the natural question: when is a subgroup of J purely indecomposable?

We first remark that a reduced torsion-free group G is isomorphic to a pure subgroup of J if and only if every p-basic subgroup of G is cyclic. The proof of this (well-known) result is accomplished by embedding G in its cotorsion completion E and an argument similar to the last half of the preceding corollary. Secondly, if J is given the usual ring structure of componentwise addition and multiplication, then every endomorphism of $(J, +)$ is a left multiplication, that is, $\alpha \in \mathrm{End}\ J$ if and only if α is a mapping of the form $x \to \pi x$ where π is a fixed element of J. Thus in case G is isomorphic to a pure subgroup of J we have the following characterization of pure indecomposability.

THEOREM. *Suppose G is a reduced torsion-free group such that each p-basic subgroup of G is cyclic. Then G is purely indecomposable if and only if, for each pure subgroup S of G, the nonzero endomorphisms of S are all monomorphisms.*

PROOF. The sufficiency is obvious, for if G contains a pure subgroup S which has a nontrivial direct decomposition then End S contains a nontrivial projection. Thus suppose G is a pure subgroup of J and that G is purely indecomposable. Let S be a pure subgroup of G and suppose $\alpha \neq 0 \in \mathrm{End}\ S$. Let A denote the kernel of α (which is pure since G is torsion-free) and B the pure closure of the image of α. Using vector notation for elements of J, let $\Lambda = [\text{primes } p \mid \text{some } a \in A \text{ has a nonzero } p\text{th coordinate}]$ and let ρ be the projection of J onto $\prod_{p \in \Lambda} I_p$. To show that $A \cap B = 0$ and that $A + B$ is a pure subgroup of G, it suffices to show that ρ is the identity when restricted to A and the zero homomorphism when restricted to B. Since I_p is an integral domain, ρ is the identity on A and zero on B if $a \cdot b = 0$ for all $a \in A$ and all $b \in B$. Let $a = \langle a_p \rangle \in A$ and $b = \langle b_p \rangle \in B$. Since S is a pure subgroup of J and since J is cotorsion, α can be extended to $\bar{\alpha} \in \mathrm{End}\ J$. Then $\bar{\alpha} = L_\pi$ for some $\pi \in J$ where L_π denotes left multiplication by $\pi = \langle \pi_p \rangle$. Therefore, $0 = L_\pi(a) = \pi a = \langle \pi_p a_p \rangle$. Thus $a_p = 0$ for each prime p such that $\pi_p \neq 0$. If $x = \langle x_p \rangle \in \mathrm{Im}\ L_\pi$, then clearly $x_p = 0$ for each prime p such that $\pi_p = 0$. It follows that $a \cdot x = 0$ if $x \in \mathrm{Im}\ \alpha$, and that there is an integer n such that $n(a \cdot b) = a(nb) = 0$ since $B = (\mathrm{Im}\ \alpha)_*$. Thus $ab = 0$ since J is torsion-free, and it follows that $A \cap B = 0$ and $A + B$ is a pure subgroup of G. Therefore, $\mathrm{Ker}\ \alpha = A = 0$ since $B = (\mathrm{Im}\ \alpha)_* \neq 0$.

In view of a result of Baer [**4**, Lemma 46.3], the above theorem is easily seen to remain true if the condition on the p-basic subgroups of G is replaced by the condition that G be homogeneous. Also, Armstrong [**2**] obtains similar results if G is p-reduced and p-purely indecomposable. However, the property that pure subgroups have no nonzero endomorphisms except monomorphisms does not, in general, characterize purely indecomposable torsion-free groups. To establish this, we consider a result of Beaumont and Pierce [**3**] which states there is a subgroup H of rank two of the two-dimensional rational vector space V such that the ring of quasi-endomorphisms $E(H)$ of H consists of all 2×2 triangular matrices over V with equal diagonal elements. Such a group H must be indecomposable and hence purely indecomposable since H is of rank two. Moreover, End H contains an endomorphism which is not a monomorphism since $E(H)$ does.

4. **The power of the collection $\mathcal{P}$ of all nonisomorphic torsion-free purely indecomposable groups.** Since the number of subgroups of J is less than or equal to $2^{\aleph}$, the corollary in §2 implies that the power of $\mathcal{P}$ is less than or equal to $2^{\aleph}$. Therefore to establish that the power of $\mathcal{P}$ is actually equal to $2^{\aleph}$, we need only show the existence of $2^{\aleph}$ nonisomorphic torsion-free purely indecomposable groups. We proceed by first proving the following lemma.

LEMMA. *Let S be a pure subgroup of J and let $I(S)$ be the collection of all subgroups of J which are isomorphic to S. Then $I(S)$ has power less than or equal to $\aleph$.*

PROOF. Let $H\in I(S)$ and let ϕ be an isomorphism of S onto H. Then $\phi\in\mathrm{Hom}\,(S, J)$. Since S is a pure subgroup of J and since J is cotorsion, ϕ can be extended to $\bar{\phi}\in\mathrm{End}\ J$. It follows that $|I(S)| \leqq |\mathrm{End}\ J| = \aleph$.

THEOREM. *There are $2^{\aleph}$ nonisomorphic pure subgroups of J which are purely indecomposable. Thus, the collection $\mathcal{P}$ of all nonisomorphic torsion-free purely indecomposable groups has power $2^{\aleph}$.*

PROOF. Since the rank of I_p is $\aleph$, let $[a_{p\lambda}]_{\lambda\in\Lambda}$ be an independent subset of I_p where $|\Lambda| = \aleph$. For each $\lambda\in\Lambda$, define $a_\lambda = \langle a_{p\lambda}\rangle \in J = \prod_p I_p$. Now the set $[a_\lambda]_{\lambda\in\Lambda}$ is an independent subset of J since for any finite subset of $[a_\lambda]_{\lambda\in\Lambda}$ the elements are componentwise independent. Therefore, for every nonvoid subset $M\subseteq\Lambda$ define $H_M = \sum_{\lambda\in M}\{a_\lambda\}$ and $G_M = (H_M)_*$. Let $h\neq 0\in H_M$ where $h = \langle h_p\rangle \in J = \prod_p I_p$. Then $h = n_1 a_{\lambda_1} + n_2 a_{\lambda_2} + \cdots + n_k a_{\lambda_k}$ where $\lambda_1, \cdots, \lambda_k \in M$ and where $n_i \neq 0$ for $i = 1, 2, \cdots, k$. Therefore $h_p = n_1 a_{p\lambda_1} + \cdots + n_k a_{p\lambda_k} \neq 0$ for each p since $[a_{p\lambda}]_{\lambda\in\Lambda}$ is an independent subset of I_p and $n_i\neq 0$ for $i = 1, 2, \cdots, k$. Thus if $g \neq 0 \in G_M$ and $g = \langle g_p\rangle$, then $g_p \neq 0$ for each p since $G_M = (H_M)_*$.

Let $S\neq 0$ be a pure subgroup of G_M and $\alpha\in\mathrm{End}\ S$ such that $\alpha(x) = 0$ for some $x\neq 0\in S$. Since S must be a pure subgroup of J and

$\alpha \in \mathrm{Hom}\,(S, J)$, then α can be extended to $L_\pi \in \mathrm{End}\, J$ where $\pi = \langle \pi_p \rangle \in J$ and L_π denotes left multiplication by π. Therefore, $0 = L_\pi(x) = \pi x = \langle \pi_p x_p \rangle$ implies $\pi_p x_p = 0$ for each p. Hence $\pi_p = 0$ for each p since we have shown that a nonzero element of G has the property that each of its components are nonzero. Thus $\pi = 0$ and therefore $\alpha = 0$. This establishes that every nonzero endomorphism of S must be a monomorphism. Hence, by our theorem in §3, G_M is purely indecomposable for every nonvoid subset M of Λ.

Let $\mathcal{G}$ denote the collection of groups G_M where M ranges over all nonvoid subsets of Λ. By the independence of $[a_\lambda]_{\lambda \in \Lambda}$ we have that $G_M = G_{M'}$ (in the set-theoretic sense) if and only if $M = M'$. Thus $\mathcal{G}$ is a collection of pure subgroups of J which are purely indecomposable, and $|\mathcal{G}| = 2^{\aleph}$. By our preceding lemma, $|I(G_M)| \leqq \aleph$ for each $G_M \in \mathcal{G}$. Since $|\mathcal{G}| = 2^{\aleph}$, it follows that there are $2^{\aleph}$ nonisomorphic pure subgroups of J which are purely indecomposable.

A slight generalization of the construction used in the preceding theorem indicates that two torsion-free purely indecomposable groups may be quite different in nature. Indeed, if N and M are two nonvoid subsets of the primes with N finite, then one may construct a torsion-free purely indecomposable group G with the following two properties:

(1) for $p \in N$, each p-basic subgroup of G has rank greater than one.

(2) For $p \in M$, $G_p^1 \neq 0$ where $G_p^1 = \bigcap_{n<\omega} p^n G$.

Finally we see that if N is nonempty and M is taken to be the set of all primes p, then such a group G as described above is neither isomorphic to a pure subgroup of J nor is it isomorphic to a subgroup of I_p for any prime p.

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