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CONVOLUTION OF L^p FUNCTIONS

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Rajagopalan and Zelazko [1] and [2] have proved that if G is a noncompact, locally compact group, $L^p(G)$ is not closed under convolution for $p>2$. In fact, although they did not prove it, the convolution of two L^p functions need not exist. This is a special case of the following:

Theorem. *If $1<p<\infty$, $1<r<\infty$, and $1/p+1/r<1$, and if G is a noncompact locally compact group, then there is an open set U in G, aud there are functions $f\in L^p(G)$, $g\in L^r(G)$, such that $f*g(y)$ is not defined for y in U.*

Proof. Let H be the subgroup of G consisting of those elements on which the modular function is 1. It is easily proved that H is closed and noncompact. Let V be a compact symmetric neighbourhood of the identity in G, and set $W=V\cdot V$. We can inductively choose a sequence x_n of elements of H such that $Wx_i\cap Wx_j=\varnothing$ for $i\neq j$ (see [2]). There is an open subset U of V such that for y in U, $y^{-1}V\cap V$ is a set of positive measure. Define $\rho>1$ so that $\rho(1/p+1/r)=1$. Choose $\epsilon_n=\pm 1$, so that the partial sums of the series $\sum\epsilon_n/n$ are everywhere dense in the real line. Define the function f on G by $f(xx_n)=1/n^{\rho/p}$ for x in V, and extend f to G by defining it to be zero outside $\bigcup Vx_n$. Likewise define h so that $h(xx_n)=\epsilon_n/n^{\rho/r}$ for x in V. Define $g(y)=h(y^{-1})$. Evidently f is in L^p and h in L^r. Since the mod-

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ular function is bounded, and bounded away from 0 on the support of h, g is in L^r. But for y in U,

$$f * g(y) = \int f(x)h(y^{-1}x)d\mu = \mu(y^{-1}V \cap V)\sum \epsilon_n/n$$

(where μ is Haar measure) which is not defined.

REMARK. If $1/p+1/r \geqq 1$, and if $f \in L^p(G)$, $g \in L^r(G)$, then it is known that $f * g$ does exist, and is in L^s, with norm at most $|f|_p |g|_r$, where $1/p+1/r-1/s=1$.

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