1. Introduction. This paper deals with the homotopy type comparison of a space with certain complexes (such as the nerve) associated with suitably "well pieced together" open covers of the space by homotopically trivial sets. The main theorem (Theorem 1) implies a slightly different version (Theorem 2) of a theorem of A. Weil [7]. The method of proof is quite different from Weil's. A simple use is made of a theorem in [6] of Alexandroff's [1] "discrete spaces" (spaces in which the intersection of any collection of open sets is open). Also use is made of a modification (Theorem 3) of a theorem of A. Dold and R. Thom [3].

It is conceivable that Theorem 1 could contribute to a positive solution of the unsolved problem: Does every compact topological manifold have the homotopy type of a finite complex? See the discussion following the statement of the theorem.

Other results to which Theorem 1 is related were obtained by J. Leray [5] and by K. Borsuk [2].

2. Notation and terminology. Suppose \( \mathcal{U} \) is an open cover of a space \( X \) (a collection of nonempty open subsets of \( X \) whose union is \( X \)). Let \( N(\mathcal{U}) \) denote the nerve of \( \mathcal{U} \). More than with \( N(\mathcal{U}) \), we shall be concerned with the following subcomplex of \( N(\mathcal{U}) \). Let \( K(\mathcal{U}) \) be the complex whose vertices are the members of \( \mathcal{U} \) and whose simplexes are the finite totally ordered subcollections of \( \mathcal{U} \) (where \( \mathcal{U} \) is partially ordered by inclusion). In general \( K(\mathcal{U}) \) does not have the same homotopy type as \( N(\mathcal{U}) \).

The open cover \( \mathcal{U} \) will be called basis-like if the intersection of any two members of \( \mathcal{U} \) is a union of members of \( \mathcal{U} \). This is equivalent to saying that \( \mathcal{U} \) is a basis for a topology on \( X \) smaller than the given one. An open cover \( \mathcal{U} \) is point-finite if each point of \( X \) is contained in only finitely many members of \( \mathcal{U} \).

A map \( f: X \to Y \) is a weak homotopy equivalence if the induced maps \( f_*: \pi_i(X, x) \to \pi_i(Y, fx) \) are isomorphisms for all \( x \in X \) and all \( i \geq 0 \). A space \( X \) is homotopically trivial if \( \pi_i(X, x) = 0 \) for all \( i \geq 0 \).

If \( K \) is an (abstract) simplicial complex, \( |K| \) denotes the underlying polyhedron with the weak topology.

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**Theorem 1.** Let $X$ be a space and let $\mathcal{U}$ be a point-finite, basis-like, open cover of $X$ by homotopically trivial sets. Then there exists a weak homotopy equivalence $f: |K(\mathcal{U})| \to X$.

**Remark.** Of course if in addition we know that $X$ has the homotopy type of a CW-complex (for example when $X$ is a topological manifold), then we may conclude from a theorem of J. H. C. Whitehead [8] that $f$ is an actual homotopy equivalence.

Thus the problem mentioned in §1 has a positive solution if the following question has an affirmative answer: Does every compact topological manifold possess a finite, basis-like, open cover by contractible sets? (In this context “contractible” is equivalent to “homotopically trivial.”) Any counterexample to this would be a counterexample to the triangulation problem, because every finite polyhedron has such an open cover: the open stars of its simplexes. However, perhaps this question could be answered affirmatively without solving the triangulation problem.

In §5, Theorem 1 will be used to derive the following variation on a theorem of Weil.

**Theorem 2 (cf. Weil [7, p. 141]).** Let $X$ be a space and let $\mathcal{V}$ be a point-finite open cover of $X$ such that the intersection of any (finite) subcollection of $\mathcal{V}$ is homotopically trivial. Then there exists a weak homotopy equivalence $|N(\mathcal{V})| \to X$.

In Weil’s version, it is assumed that $X \times X \times [0, 1]$ is normal, that $\mathcal{V}$ is locally finite, and that the intersections of subcollections of $\mathcal{V}$ are solid; and it is concluded that there is an actual homotopy equivalence $|N(\mathcal{V})| \to X$. If we replace the first of these assumptions by the assumption that $X$ is separable metric, then we can derive Weil’s conclusion, because then by a theorem of O. Hanner [4, p. 392], $X$ is an ANR, so that Whitehead’s theorem applies.

There are simple examples to which Theorem 1 applies, but where $|N(\mathcal{U})|$ does not have the (weak) homotopy type of $X$.

The main tool in proving Theorem 1 is the following theorem used in [6]. This theorem follows from a modification of the proof of Satz 2.2 of Dold and Thom [3]. For details on the necessary modification, see [6].

**Theorem 3.** Suppose $p$ is a map of a space $X$ into a space $Y$ and suppose there exists a basis-like open cover $\mathcal{W}$ of $Y$ satisfying the following condition: For each $W \in \mathcal{W}$, the restriction $p|_{p^{-1}(W)}: p^{-1}(W) \to W$ is a weak homotopy equivalence. Then $p$ itself is a weak homotopy equivalence.
4. Proof of Theorem 1. First let us state the following

Lemma 1. If in the diagram

\[ \begin{align*}
\begin{array}{c}
X \\
\downarrow f \\
K \\
\downarrow g \\
Y
\end{array}
\end{align*} \]

is a CW-complex and \( p \) is a weak homotopy equivalence, then \( f \) can be found making the diagram homotopy commutative. Hence if \( g \) is also a weak homotopy equivalence, then so is \( f \).

The proof, which is simple and involves the mapping cylinder of \( p \), appears as a part of an argument on page 244 of [3].

Now suppose \( X \) and \( \mathcal{U} \) are as in the statement of Theorem 1. Using the fact that \( \mathcal{U} \) is partially ordered by inclusion and using the modification in [6] of Alexandroff's [1] procedure, we make \( \mathcal{U} \) into a topological space as follows. For each \( U \in \mathcal{U} \), let \( [U] = \{ V \in \mathcal{U} : V \subseteq U \} \). Then as a basis for the required topology we take the collection \( \{ [U] : U \in \mathcal{U} \} \).

As in [6], we define a map \( g : \left| K(\mathcal{U}) \right| \to \mathcal{U} \) as follows: If \( x \in \left| K(\mathcal{U}) \right| \), then let \( (U_0, \cdots, U_n) \) be the unique open simplex of \( \left| K(\mathcal{U}) \right| \) to which \( x \) belongs, with \( U_0 \subseteq \cdots \subseteq U_n \). Then \( g(x) = U_0 \). In [6] it is shown that \( g \) is continuous and, by use of Theorem 3, that \( g \) is a weak homotopy equivalence.

Next we define a map \( p : X \to \mathcal{U} \). For each \( x \in X \) let \( p(x) \) be the smallest member of \( \mathcal{U} \) containing \( x \). This exists since \( \mathcal{U} \) is point-finite and basis-like. The proof of the following lemma is straightforward.

Lemma 2. For each \( U \in \mathcal{U} \), \( p^{-1}([U]) = U \).

Since the sets \( [U] \) form (by definition) a basis for the space \( \mathcal{U} \), this implies that \( p \) is continuous.

Now we wish to apply Theorem 3 to show that \( p \) is a weak homotopy equivalence. For the basis-like open cover of \( \mathcal{U} \) we take simply the basis \( \mathcal{W} = \{ [U] : U \in \mathcal{U} \} \). It is shown in [6] that each \( [U] \) is a contractible subset of \( \mathcal{U} \). By Lemma 2, \( p^{-1}([U]) = U \); and \( p \mid U : U \to [U] \) is trivially a weak homotopy equivalence, since \( U \) is assumed to be homotopically trivial. Hence Theorem 3 implies that \( p \) is a weak homotopy equivalence. Now we apply Lemma 1 to obtain a weak homotopy equivalence \( f : \left| K(\mathcal{U}) \right| \to X \).

5. Proof of Theorem 2. Let \( \mathcal{U} \) be the collection of all nonempty intersections of (finite) subcollections of \( \mathcal{V} \). Clearly \( \mathcal{U} \) satisfies the conditions of Theorem 1, so that we get a weak homotopy equiva-
lence \( |K(\mathcal{U})| \to X \). From the facts that \( \mathcal{V} \subset \mathcal{U} \) and that \( \mathcal{U} \) is a refinement of \( \mathcal{V} \), it is easy to see that \( |N(\mathcal{V})| \) is a deformation retract of \( |N(\mathcal{U})| \). (More generally, if two covers of a space refine each other, then their nerves have the same homotopy type.) Hence the proof of the theorem is completed by the following

**Lemma 3.** Let \( \mathcal{U} \) be a cover of a space with the property that the intersection of any finite subcollection of \( \mathcal{U} \) is either empty or a member of \( \mathcal{U} \). Then \( |K(\mathcal{U})| \) is a deformation retract of \( |N(\mathcal{U})| \).

**Proof.** Abbreviate \( K = K(\mathcal{U}) \), \( N = N(\mathcal{U}) \), and let \( N' \) be the first barycentric subdivision of \( N \). Let us define a simplicial map \( \phi : N' \to K \) as follows. The vertices of \( N' \) are the barycenters \( b(\sigma) \) of the simplexes \( \sigma \) of \( N \). Define \( \phi \) on these vertices by the equation

\[
\phi(b(\sigma)) = \text{Carrier } \sigma
\]

(By assumption Carrier \( \sigma \), the intersection of the vertices of \( \sigma \), being nonempty, is a member of \( \mathcal{U} \), that is, a vertex of \( K \).) Any simplex of \( N' \) is spanned by vertices \( b(\sigma_0), \ldots, b(\sigma_n) \), where \( \sigma_0 \subset \cdots \subset \sigma_n \). Then Carrier \( \sigma_0 \subset \cdots \subset \text{Carrier } \sigma_n \), so that these vertices span a simplex of \( K \). Thus \( \phi \) is simplicial.

Let us show that \( \phi \), as a map of \( |N| = |N'| \) into \( |K| \), is a deformation retraction. Clearly it is a retraction. It suffices then to show that for each simplex \( \tau \) of \( N' \), both \( \tau \) and \( \phi(\tau) \) are subsets of some simplex \( \sigma \) of \( N \). Let the vertices of \( \tau \) be \( b(\sigma_0), \ldots, b(\sigma_n) \), where \( \sigma_0 \subset \cdots \subset \sigma_n \), and let the vertices of \( \sigma_n \) be \( U_0, \ldots, U_\tau \). We can take \( \sigma \) to be the simplex of \( N \) spanned by the vertices

\[
\{ U_0, \ldots, U_\tau, \text{Carrier } \sigma_0, \ldots, \text{Carrier } \sigma_n \}.
\]

**References**


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