HOMOTOPY TYPE COMPARISON OF A SPACE WITH COMPLEXES ASSOCIATED WITH ITS OPEN COVERS

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1. Introduction. This paper deals with the homotopy type comparison of a space with certain complexes (such as the nerve) associated with suitably "well pieced together" open covers of the space by homotopically trivial sets. The main theorem (Theorem 1) implies a slightly different version (Theorem 2) of a theorem of A. Weil [7]. The method of proof is quite different from Weil's. A simple use is made of a theorem in [6] of Alexandroff's [1] "discrete spaces" (spaces in which the intersection of any collection of open sets is open). Also use is made of a modification (Theorem 3) of a theorem of A. Dold and R. Thom [3].

It is conceivable that Theorem 1 could contribute to a positive solution of the unsolved problem: Does every compact topological manifold have the homotopy type of a finite complex? See the discussion following the statement of the theorem.

Other results to which Theorem 1 is related were obtained by J. Leray [5] and by K. Borsuk [2].

2. Notation and terminology. Suppose $\mathcal{U}$ is an open cover of a space $X$ (a collection of nonempty open subsets of $X$ whose union is $X$). Let $N(\mathcal{U})$ denote the nerve of $\mathcal{U}$. More than with $N(\mathcal{U})$, we shall be concerned with the following subcomplex of $N(\mathcal{U})$. Let $K(\mathcal{U})$ be the complex whose vertices are the members of $\mathcal{U}$ and whose simplexes are the finite totally ordered subcollections of $\mathcal{U}$ (where $\mathcal{U}$ is partially ordered by inclusion). In general $K(\mathcal{U})$ does not have the same homotopy type as $N(\mathcal{U})$.

The open cover $\mathcal{U}$ will be called basis-like if the intersection of any two members of $\mathcal{U}$ is a union of members of $\mathcal{U}$. This is equivalent to saying that $\mathcal{U}$ is a basis for a topology on $X$ smaller than the given one. An open cover $\mathcal{U}$ is point-finite if each point of $X$ is contained in only finitely many members of $\mathcal{U}$.

A map $f: X \to Y$ is a weak homotopy equivalence if the induced maps $f_*: \pi_i(X, x) \to \pi_i(Y, f x)$ are isomorphisms for all $x \in X$ and all $i \geq 0$. A space $X$ is homotopically trivial if $\pi_i(X, x) = 0$ for all $i \geq 0$.

If $K$ is an (abstract) simplicial complex, $|K|$ denotes the underlying polyhedron with the weak topology.

Theorem 1. Let \( X \) be a space and let \( \mathcal{U} \) be a point-finite, basis-like, open cover of \( X \) by homotopically trivial sets. Then there exists a weak homotopy equivalence \( f: |K(\mathcal{U})| \rightarrow X \).

Remark. Of course if in addition we know that \( X \) has the homotopy type of a CW-complex (for example when \( X \) is a topological manifold), then we may conclude from a theorem of J. H. C. Whitehead [8] that \( f \) is an actual homotopy equivalence.

Thus the problem mentioned in §1 has a positive solution if the following question has an affirmative answer: Does every compact topological manifold possess a finite, basis-like, open cover by contractible sets? (In this context “contractible” is equivalent to “homotopically trivial.”) Any counterexample to this would be a counterexample to the triangulation problem, because every finite polyhedron has such an open cover: the open stars of its simplexes. However, perhaps this question could be answered affirmatively without solving the triangulation problem.

In §5, Theorem 1 will be used to derive the following variation on a theorem of Weil.

Theorem 2 (cf. Weil [7, p. 141]). Let \( X \) be a space and let \( \mathcal{V} \) be a point-finite open cover of \( X \) such that the intersection of any (finite) subcollection of \( \mathcal{V} \) is homotopically trivial. Then there exists a weak homotopy equivalence \( |N(\mathcal{V})| \rightarrow X \).

In Weil’s version, it is assumed that \( X \times X \times [0, 1] \) is normal, that \( \mathcal{V} \) is locally finite, and that the intersections of subcollections of \( \mathcal{V} \) are solid; and it is concluded that there is an actual homotopy equivalence \( |N(\mathcal{V})| \rightarrow X \). If we replace the first of these assumptions by the assumption that \( X \) is separable metric, then we can derive Weil’s conclusion, because then by a theorem of O. Hanner [4, p. 392], \( X \) is an ANR, so that Whitehead’s theorem applies.

There are simple examples to which Theorem 1 applies, but where \( |N(\mathcal{U})| \) does not have the (weak) homotopy type of \( X \).

The main tool in proving Theorem 1 is the following theorem used in [6]. This theorem follows from a modification of the proof of Satz 2.2 of Dold and Thom [3]. For details on the necessary modification, see [6].

Theorem 3. Suppose \( p \) is a map of a space \( X \) into a space \( Y \) and suppose there exists a basis-like open cover \( \mathcal{W} \) of \( Y \) satisfying the following condition: For each \( W \in \mathcal{W} \), the restriction \( p|p^{-1}(W): p^{-1}(W) \rightarrow W \) is a weak homotopy equivalence. Then \( p \) itself is a weak homotopy equivalence.
4. Proof of Theorem 1. First let us state the following

**Lemma 1.** If in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{g} & & \\
K & \xrightarrow{p} & Y
\end{array}
\]

\(K\) is a CW-complex and \(p\) is a weak homotopy equivalence, then \(f\) can be found making the diagram homotopy commutative. Hence if \(g\) is also a weak homotopy equivalence, then so is \(f\).

The proof, which is simple and involves the mapping cylinder of \(p\), appears as a part of an argument on page 244 of [3].

Now suppose \(X\) and \(\mathcal{U}\) are as in the statement of Theorem 1. Using the fact that \(\mathcal{U}\) is partially ordered by inclusion and using the modification in [6] of Alexandroff’s [1] procedure, we make \(\mathcal{U}\) into a topological space as follows. For each \(U \in \mathcal{U}\), let \([U]\) = \(\{ V \in \mathcal{U} : V \subset U \}\). Then as a basis for the required topology we take the collection \(\{ [U] : U \in \mathcal{U} \}\).

As in [6], we define a map \(g : |K\langle\mathcal{U}\rangle| \to \mathcal{U}\) as follows: If \(x \in |K\langle\mathcal{U}\rangle|\), then let \((U_0, \ldots, U_n)\) be the unique open simplex of \(|K\langle\mathcal{U}\rangle|\) to which \(x\) belongs, with \(U_0 \subset \cdots \subset U_n\). Then \(g(x) = U_0\). In [6] it is shown that \(g\) is continuous and, by use of Theorem 3, that \(g\) is a weak homotopy equivalence.

Next we define a map \(p : X \to \mathcal{U}\). For each \(x \in X\) let \(p(x)\) be the smallest member of \(\mathcal{U}\) containing \(x\). This exists since \(\mathcal{U}\) is point-finite and basis-like. The proof of the following lemma is straightforward.

**Lemma 2.** For each \(U \in \mathcal{U}\), \(p^{-1}([U]) = U\).

Since the sets \([U]\) form (by definition) a basis for the space \(\mathcal{U}\), this implies that \(p\) is continuous.

Now we wish to apply Theorem 3 to show that \(p\) is a weak homotopy equivalence. For the basis-like open cover of \(\mathcal{U}\) we take simply the basis \(\mathcal{W} = \{ [U] : U \in \mathcal{U} \}\). It is shown in [6] that each \([U]\) is a contractible subset of \(\mathcal{U}\). By Lemma 2, \(p^{-1}([U]) = U\); and \(p \mid U : U \to [U]\) is trivially a weak homotopy equivalence, since \(U\) is assumed to be homotopically trivial. Hence Theorem 3 implies that \(p\) is a weak homotopy equivalence. Now we apply Lemma 1 to obtain a weak homotopy equivalence \(f : |K\langle\mathcal{U}\rangle| \to X\).

5. Proof of Theorem 2. Let \(\mathcal{U}\) be the collection of all nonempty intersections of (finite) subcollections of \(\mathcal{V}\). Clearly \(\mathcal{U}\) satisfies the conditions of Theorem 1, so that we get a weak homotopy equiva-
lence $|K(\mathcal{U})| \to X$. From the facts that $\mathcal{V} \subset \mathcal{U}$ and that $\mathcal{U}$ is a refinement of $\mathcal{V}$, it is easy to see that $|N(\mathcal{V})|$ is a deformation retract of $|N(\mathcal{U})|$. (More generally, if two covers of a space refine each other, then their nerves have the same homotopy type.) Hence the proof of the theorem is completed by the following

**Lemma 3.** Let $\mathcal{U}$ be a cover of a space with the property that the intersection of any finite subcollection of $\mathcal{U}$ is either empty or a member of $\mathcal{U}$. Then $|K(\mathcal{U})|$ is a deformation retract of $|N(\mathcal{U})|$.

**Proof.** Abbreviate $K = K(\mathcal{U})$, $N = N(\mathcal{U})$, and let $N'$ be the first barycentric subdivision of $N$. Let us define a simplicial map $\phi: N' \to K$ as follows. The vertices of $N'$ are the barycenters $b(\sigma)$ of the simplexes $\sigma$ of $N$. Define $\phi$ on these vertices by the equation

$$\phi(b(\sigma)) = \text{Carrier } \sigma$$

(By assumption Carrier $\sigma$, the intersection of the vertices of $\sigma$, being nonempty, is a member of $\mathcal{U}$, that is, a vertex of $K$.) Any simplex of $N'$ is spanned by vertices $b(\sigma_0), \ldots, b(\sigma_n)$, where $\sigma_0 \subset \cdots \subset \sigma_n$. Then Carrier $\sigma_0 \supset \cdots \supset \text{Carrier } \sigma_n$, so that these vertices span a simplex of $K$. Thus $\phi$ is simplicial.

Let us show that $\phi$, as a map of $|N| = |N'|$ into $|K|$, is a deformation retraction. Clearly it is a retraction. It suffices then to show that for each simplex $\tau$ of $N'$, both $\tau$ and $\phi(\tau)$ are subsets of some simplex $\sigma$ of $N$. Let the vertices of $\tau$ be $b(\sigma_0), \ldots, b(\sigma_n)$, where $\sigma_0 \subset \cdots \subset \sigma_n$, and let the vertices of $\sigma_n$ be $U_0, \ldots, U_r$. We can take $\sigma$ to be the simplex of $N$ spanned by the vertices

$$\{U_0, \ldots, U_r, \text{ Carrier } \sigma_0, \ldots, \text{ Carrier } \sigma_n\}.$$

**References**