ON THE LEBESGUE DECOMPOSITION THEOREM

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1. **Introduction.** All measures in this paper are measures on a σ-ring $\mathcal{S}$ of subsets of X. We say that ν is *absolutely continuous* with respect to μ, denoted $\nu \ll \mu$, if $\nu(E)=0$ for each measurable E (i.e., each $E \in \mathcal{S}$) such that $\mu(E)=0$ [**3**, p. 124]. A set $A \subset X$ is *locally measurable* if $A \cap E$ is measurable for each measurable set E [**1**, p. 35]. We say that ν is *singular* with respect to μ, denoted $\nu \perp \mu$, if there exists locally measurable A such that $\nu(E \cap A)=0=\mu(E-A)$ for each $E \in \mathcal{S}$ (cf. [**3**, p. 126]). Of course, if $\nu \perp \mu$, then $\mu \perp \nu$. Finally, we say that ν is *S-singular* with respect to μ, denoted $\nu S \mu$, if given $E \in \mathcal{S}$, there exists measurable $F \subset E$ such that $\nu(E)=\nu(F)$ and $\mu(F)=0$.

The purpose of this paper is to give a proof of the Lebesgue decomposition theorem as stated in Theorem 2.1. Our statement is somewhat more general than usual statements (cf. [**3**, 32.C]) in that no restrictions are placed on the measures in question. This generality is achieved by using the weaker S-singularity rather than singularity and by sacrificing uniqueness if the decomposed measure is not σ-finite. In §3 we investigate some of the properties of S-singularity and give a condition (Theorem 3.3) under which S-singularity implies singularity.

2. **The Lebesgue decomposition theorem.** The following lemma will be used implicitly in the proof of Theorem 2.1.

LEMMA. *Suppose λ is a measure on $\mathcal{S}$ and $\mathcal{M}$ is a subfamily of $\mathcal{S}$ such that $\mathcal{M}$ is closed under countable unions. Define a set function λ' on $\mathcal{S}$ by*

$$\lambda'(E)=\mathrm{LUB}\{\lambda(E \cap M) : M \in \mathcal{M}\}, \quad \textit{for each } E \in \mathcal{S}.$$

Then:

(1) *For each $E \in \mathcal{S}$, there exists $M \in \mathcal{M}$ such that $\lambda'(E)=\lambda(E \cap M)$.*
(2) *$\lambda'(E) \leqq \lambda(E)$ for each $E \in \mathcal{S}$, and $\lambda'(M)=\lambda(M)$ for each $M \in \mathcal{M}$.*
(3) *λ' is a measure on $\mathcal{S}$.*

PROOF. (1) follows from the fact that $\mathcal{M}$ is closed under countable unions. (2) follows immediately from (1). To prove (3), we notice that the set functions λ_M defined by $\lambda_M(E)=\lambda(E \cap M)$, $E \in \mathcal{S}$, are measures on $\mathcal{S}$. Moreover, they are increasingly directed in the obvious sense, so that their least upper bound, namely λ', is a measure [**1**, 10.1]. A proof of (3) can also be based directly on (1).

Received by the editors April 8, 1966.

THEOREM 2.1. *If μ and ν are measures on $\mathcal{S}$, then there exist $\nu_0 \ll \mu$ and $\nu_1 S \mu$ such that $\nu = \nu_0 + \nu_1$. Always ν_1 is unique. Hence, if ν is σ-finite, then ν_0 is unique also.*

PROOF. Let $\mathfrak{N} = \{N \in \mathcal{S}: \mu(N) = 0\}$. Clearly $\mathfrak{N}$ is closed under countable unions. Define a measure ν_1 on $\mathcal{S}$ by

$$\nu_1(E) = \text{LUB}\{\nu(E \cap N): N \in \mathfrak{N}\}.$$

If $E \in \mathcal{S}$, then $\nu_1(E) = \nu_1(E \cap N)$ for some $N \in \mathfrak{N}$, and $\mu(E \cap N) = 0$. Hence, $\nu_1 S \mu$.

Let $\mathfrak{M} = \{M \in \mathcal{S}: \nu_1(M) = 0\}$. Clearly $\mathfrak{M}$ is closed under countable unions. Define a measure ν_0 on $\mathcal{S}$ by

$$\nu_0(E) = \text{LUB}\{\nu(E \cap M): M \in \mathfrak{M}\}.$$

If $E \in \mathcal{S}$, then $\nu_0(E) = \nu(E \cap M)$ for some $M \in \mathfrak{M}$. If, moreover, $\mu(E) = 0$, then $E \cap M \in \mathfrak{N}$, so that $\nu_0(E) = \nu(E \cap M) = \nu_1(E \cap M) = 0$. Hence, $\nu_0 \ll \mu$.

We show that $\nu = \nu_0 + \nu_1$. Suppose $E \in \mathcal{S}$. If $\nu_1(E) = \infty$, then clearly $\nu(E) = \infty$, so that we are done. On the other hand, suppose $\nu_1(E) < \infty$. There exists $N \in \mathfrak{N}$ such that $\nu_1(E) = \nu(E \cap N) = \nu_1(E \cap N)$. In this case, $\nu_1(E - N) = 0$, so that $E - N \in \mathfrak{M}$. Hence, $\nu(E - N) = \nu_0(E - N)$. Since $\mu(E \cap N) = 0$, we have $\nu_0(E \cap N) = 0$, so that $\nu(E - N) = \nu_0(E)$. Hence, $\nu(E) = \nu(E - N) + \nu(E \cap N) = \nu_0(E) + \nu_1(E)$.

Now suppose $\nu_0 + \nu_1$ and $\nu_2 + \nu_3$ are two such decompositions of ν. If $E \in \mathcal{S}$, we wish to show that $\nu_1(E) = \nu_3(E)$. Now, by assumption, there exists $F_1 \subset E$ such that $\nu_1(E) = \nu_1(F_1)$ and $\mu(F_1) = 0$, and there exists $F_3 \subset E$ such that $\nu_3(E) = \nu_3(F_3)$ and $\mu(F_3) = 0$. Letting $F = F_1 \cup F_3$, we have $\mu(F) = 0$, so that $\nu_0(F) = 0$ and $\nu_2(F) = 0$. Hence, $\nu_1(F) = \nu_3(F)$. It is easy to see that $\nu_1(E) = \nu_1(F)$ and $\nu_3(E) = \nu_3(F)$, so that $\nu_1(E) = \nu_3(E)$, which completes the proof.

ν_0 need not be unique, even if μ is a totally finite measure. For, let $\mathcal{S}$ be the Borel sets of the unit interval, μ be Lebesgue measure on $\mathcal{S}$, and ν be counting measure on $\mathcal{S}$. Then $\nu = 0 + \nu$ and $\nu = \mu + \nu$ are distinct decompositions for ν with respect to μ. We notice, incidentally, that $\nu S \mu$; but it is false that $\mu S \nu$.

3. On S-singularity and singularity. We list some properties of S-singularity, omitting their easy proofs. All of these properties hold if S-singularity is replaced by singularity.

1. $\nu S \nu$ if and only if $\nu = 0$.
2. If $\nu S \mu$ and $\lambda \ll \mu$, then $\nu S \lambda$.
3. If $\nu S \mu$ and $\nu \ll \mu$, then $\nu = 0$.

4a. If $\nu S \mu$ and $\lambda S \mu$, then $(\nu + \lambda) S \mu$.

b. If $\nu S\mu$ and $\nu S\lambda$, then $\nu S(\mu+\lambda)$.

5a. If $\nu \ll \mu+\lambda$ and $\nu S\lambda$, then $\nu \ll \mu$.

b. If $\nu \leqq \mu+\lambda$ and $\nu S\lambda$, then $\nu \leqq \mu$.

6. If $\nu_n \to \nu$ (i.e., $\nu_n(E) \to \nu(E)$ for each $E \in \mathcal{S}$) and $\nu_n S\mu$ for each n, then $\nu S\mu$.

THEOREM 3.1. *If $\{\nu_\alpha\}$ is an increasingly directed family of measures on $\mathcal{S}$, $\nu = \text{LUB } \nu_\alpha$, and $\nu_\alpha S\mu$ for each α, then $\nu S\mu$.*

PROOF. Of course ν is a measure [1, 10.1]. Now given $E \in \mathcal{S}$, choose a "subsequence" $\{\alpha(n)\}$ of the α's such that $\nu_{\alpha(n)}(E) \uparrow \nu(E)$. For each n, choose measurable $F_n \subset E$ such that $\nu_{\alpha(n)}(E) = \nu_{\alpha(n)}(F_n)$ and $\mu(F_n) = 0$. Let $F = \bigcup F_n$. Then $\nu_{\alpha(n)}(E) = \nu_{\alpha(n)}(F)$ for each n, and $\mu(F) = 0$. Clearly,

$$\nu(E) = \text{LUB } \nu_{\alpha(n)}(E) = \text{LUB } \nu_{\alpha(n)}(F) \leqq \nu(F) \leqq \nu(E),$$

so that $\nu(E) = \nu(F)$, where $\mu(F) = 0$.

Theorem 3.1 indicates an advantage of S-singularity over singularity since the theorem is false if S-singularity is replaced by singularity. For, let $\mathcal{S}$ be the Borel sets of the real line, μ be Lebesgue measure on $\mathcal{S}$, and ν be counting measure on $\mathcal{S}$. If I is a finite subset of the real line, let $\nu_I(E)$ be the number of points in $E \cap I$, for each $E \in \mathcal{S}$. The measures ν_I are increasingly directed in the obvious sense, $\nu = \text{LUB } \nu_I$, $\nu_I \perp \mu$ for each finite I, but it is false that $\nu \perp \mu$. Incidentally, $\mu S\nu_I$ for each such I, but it is false that $\mu S\nu$.

In view of property 6 and Theorem 3.1, we may ask the following question. If $\nu_\alpha(E) \to \nu(E)$ for each $E \in \mathcal{S}$ and $\nu_\alpha S\mu$ for each α, does it follow that $\nu S\mu$? The answer is no, as the following example shows. Let X be the set of ordinals less than the first uncountable ordinal, and let $\mathcal{S}$ be the class of countable sets in X and their complements. If $\alpha \in X$, let $\nu_\alpha(E) = 1$ if $\alpha \in E$ and 0 otherwise. Let $\nu(E) = 0$ if E is countable and 1 otherwise. It is easy to see that $\nu_\alpha \to \nu$ under the obvious ordering and that $\nu_\alpha S\nu$ for each $\alpha \in X$. But it is false that $\nu S\nu$.

We saw at the end of §2 that $\nu S\mu$ does not imply that $\mu S\nu$. We now give a condition under which S-singularity is symmetric.

THEOREM 3.2. *If ν is σ-finite and $\nu S\mu$, then given $E \in \mathcal{S}$, there exists measurable $F \subset E$ such that $\nu(E-F) = 0$ and $\mu(F) = 0$. Hence, $\mu S\nu$ in this case.*

PROOF. If $\nu(E) < \infty$, then the conclusion is obvious. In the general case, we write $E = \bigcup E_n$, where $\nu(E_n) < \infty$. For each n, choose measurable $F_n \subset E_n$ such that $\nu(E_n - F_n) = 0$ and $\mu(F_n) = 0$. If $F = \bigcup F_n$, then $\mu(F) = 0$ and $\nu(E - F) \leqq \nu(\bigcup (E_n - F_n)) \leqq \sum \nu(E_n - F_n) = 0$, so that $\nu(E - F) = 0$. Since $\mu(E) = \mu(E - F)$, and $\nu(E - F) = 0$, we have $\mu S \nu$ in this case.

It is easily seen that $\nu \perp \mu$ implies $\nu S \mu$ and $\mu S \nu$. However, the converse need not hold. Consider, for example, the space of [**3**, Exercise 31.9] and let $\mu(E)$ be the number of horizontal lines on which E is full and $\nu(E)$ be the number of vertical lines on which E is full. If there exists locally measurable A such that $\nu(E \cap A) = 0 = \mu(E - A)$ for each $E \in \mathcal{S}$, then A is full on horizontal lines and countable on vertical lines, which is impossible. Hence, it is false that $\nu \perp \mu$, although it is easy to see that $\nu S \mu$ and $\mu S \nu$.

We now consider a condition under which S-singularity implies singularity. We shall say that a measure λ is *strongly σ-finite* if there exists a disjoint class of measurable sets F_α, $\lambda(F_\alpha) < \infty$, such that every member of $\mathcal{S}$ intersects only a countable number of the F_α, and $\lambda(E) = \sum \lambda(E \cap F_\alpha)$. We refer to the family $\{F_\alpha\}$ as a class of *finite islands* for λ. Of course, λ is necessarily σ-finite.

The following are examples of strongly σ-finite measures:

1. Any finite measure.
2. Any σ-finite measure on a σ-algebra.
3. Any σ-finite measure λ such that $\lambda \ll \mu$, where μ is a strongly σ-finite measure.
4. The sum of any two strongly σ-finite measures.
5. Any regular Borel measure λ (as defined in [**3**, pp. 223–224]). For, let $\{F_\alpha\}$ be a maximal disjoint family of compact sets F_α such that for each α and each open set U, either U is disjoint from F_α or $\lambda(U \cap F_\alpha) > 0$ [**2**, 4.14.9]. The family $\{F_\alpha\}$ is indeed a class of finite islands for λ. For, a bounded open set (hence any open Borel set and hence any Borel set) clearly intersects only a countable number of the F_α's, and $\lambda(E) = \sum \lambda(E \cap F_\alpha)$ by the maximality of the family $\{F_\alpha\}$.

THEOREM 3.3. *Suppose $\nu S \mu$. If ν is strongly σ-finite (or if ν is σ-finite and μ is strongly σ-finite), then $\mu \perp \nu$ and $\nu \perp \mu$.*

PROOF. Suppose ν is a strongly σ-finite measure, and let $\{F_\alpha\}$ be a class of finite islands for ν. By Theorem 3.2, there exist measurable $G_\alpha \subset F_\alpha$ such that $\nu(F_\alpha - G_\alpha) = 0$ and $\mu(G_\alpha) = 0$. Let $A = \bigcup G_\alpha$. Then, clearly $\mu(E \cap A) = 0$ for all $E \in \mathcal{S}$. Moreover, if $E \in \mathcal{S}$, then

$$\nu(E - A) = \sum \nu((E - A) \cap F_\alpha) = \sum \nu(E \cap (F_\alpha - G_\alpha)) = 0.$$

Hence, $\mu \perp \nu$ and $\nu \perp \mu$. The second case follows from the first, since we have $\mu S \nu$ by Theorem 3.2.

The following form of the Lebesgue decomposition theorem is now obvious.

THEOREM 3.4. *If ν is a strongly σ-finite measure (or if ν is σ-finite and μ is strongly σ-finite), then there exist unique ν_0 and ν_1 such that $\nu = \nu_0 + \nu_1$, where $\nu_0 \ll \mu$ and $\nu_1 \perp \mu$.*

4. **Remarks on the Radon-Nikodym theorem.** We say that a nonnegative extended real-valued function f is *locally measurable* if $f^{-1}[M]$ is a locally measurable set for each Borel set M and for $M = \{\infty\}$. Now, since the condition of strongly σ-finiteness is similar (though not equivalent) to other conditions used in a Radon-Nikodym theorem (cf. [**3**, Exercise 31.10 and **4**, 11.24]), it is reasonable that strongly σ-finiteness may play a role in such a theorem. The facts are as follows: If $\nu \ll \mu$, where μ is strongly σ-finite (or ν is strongly σ-finite and μ is σ-finite), then there exists a nonnegative locally measurable function f such that $\nu(E) = \int_E f d\mu$ for each measurable set E. If f and g are two such functions, then the intersection of $\{x: f(x) \neq g(x)\}$ with any measurable set has μ-measure 0. These statements follow from the Radon-Nikodym theorem [**1**, 51.1] for finite μ and ν.

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