THE MOMENTS OF RECURRENCE TIME

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In connection with Poincaré's recurrence theorem Kac [1] obtained the mean of the recurrence time (formula (3) below) and the author [2] gave a very simple proof of this result. Recently Blum and Rosenblatt [3] obtained the higher moments (formula (2) below). In the present note we obtain both results by an exceedingly simple and perspicuous argument. This note is entirely self-contained.

Let \( \Omega \) be a point set, \( m \) a probability measure on \( \Omega \), and \( T \) a one-to-one ergodic measure-preserving transformation of \( \Omega \) into itself. Let \( A \subset \Omega \) be such that \( m(A) > 0 \). For any point \( a \) in \( \Omega \) let \( n(a) \) be the smallest positive integer such that \( T^n a \in A \); if no such integer exists let \( n(a) = \infty \). Define \( A_k = \{ a \in A \mid n(a) = k \} \), \( \overline{A} = \Omega - A \), and \( \Gamma_k = \{ a \in \overline{A} \mid n(a) = k \} \). Borrowing the notation of [3] we will define

\[
(1) \quad p_n = m\{ \Gamma_n \cup \Gamma_{n+1} \cup \cdots \},
\]

for \( n \geq 1 \). We will also make use of the usual combinatorial symbol \((k)_j = k(k-1) \cdots (k-j+1)\) for \( k \) and \( j \) positive integers, with \((k)_0 = 1\).

Our object will be to prove that

\[
(2) \quad D_j = \int_A [n(a)]_j dm = j(j - 1) \sum_{k=j-2}^{\infty} (k)_{j-2} p_{k+1}
\]

References

for \( j \geq 2 \), the result of [3]. The result of [1] (also proved in [2]) is

\[(3) \quad D_1 = 1.\]

By Poincaré’s recurrence theorem (e.g., [2]; ergodicity of \( T \) is not required) one has that \( m(A_\infty) = 0 \). The ergodicity of \( T \) implies that \( m(\Gamma_\infty) = 0 \).

The basic formula of our argument will be

\[(4) \quad T(A_k \cup \Gamma_k) = \Gamma_{k-1}\]

for \( k \geq 2 \); it is so obvious as not to require proof. Using (4) repeatedly for \( k = n+1, n+2, \ldots \) we obtain that

\[(5) \quad m(\Gamma_n) = \sum_{k=n+1}^{\infty} m(A_k), \quad n \geq 1;\]

\[(6) \quad p_n = \sum_{k=n+1}^{\infty} (k-n)m(A_k), \quad n \geq 1.\]

Thus

\[(7) \quad p_1 = m(A_2) + 2m(A_3) + 3m(A_4) + \cdots = D_1 - \sum_{k=1}^{\infty} m(A_k) = D_1 - m(A).\]

Obviously

\[(8) \quad p_1 = m(\overline{A}) = 1 - m(A),\]

so that (7) and (8) prove (3).

Using (6) in the right member of (2) we obtain that the coefficient of \( m(A_k), k \geq j \), in the right member of (2) is

\[(9) \quad j(j-1) \left[ \sum_{i=1}^{k-j+1} i(k-i-1)(j-2) \right],\]

which is easily shown (e.g., by induction) to equal \( \langle k \rangle_j \). This proves (2).

**References**