REPRESENTABILITY OF PARTIAL RECURSIVE FUNCTIONS IN FORMAL THEORIES

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1. Introduction. Ehrenfeucht and Feferman have shown [**1**] that all recursively enumerable sets X of natural numbers are "representable" in any consistent recursively enumerable theory S in which all recursive functions are definable (in the sense of Tarski-Mostowski-Robinson [**4**]) and which has a formula $x_1 \leqq x_2$ satisfying conditions (i), (ii) below for each natural number n:

(i) $\vdash_S x_1 \leqq \bar{n} \equiv x_1 = \bar{0} \vee x_1 = \bar{1} \vee \cdots \vee x_1 = \bar{n}$,

(ii) $\vdash_S x_1 \leqq \bar{n} \vee \bar{n} \leqq x_1$.

(Here $\bar{n}$ is the (closed) numerical term of S corresponding to n, i.e. Δ_n of [**4**, p. 44].) (By a construction of Cobham (see [**3**, p. 121] for details), (ii) is redundant in the presence of (i) and the definability in S of the successor function.) That is, for such an X, there is a formula $\Phi(x_1)$ of S (with exactly one free variable x_1) such that for every n,

$$n \in X \Leftrightarrow \vdash_S \Phi(\bar{n}).$$

The argument is to show that there is some creative set C representable in S, from which the result follows by the reducibility of X to C by some recursive function (Myhill). Shepherdson has obtained the result [**3**] more directly by an elegant adaptation of Rosser-type arguments, much as Bernays obtained results of Myhill on theories. In [**2**] Ritchie and Young show that in every consistent recursively enumerable extension S of R. M. Robinson's system R ([**4**, pp. 52–53]), all partial recursive functions ϕ are "strongly representable." That is, for such a ϕ, there is a formula $\Phi(x_1, x_2)$ of S such that for all m, n,

(iii) $\phi(m) = n \Leftrightarrow \vdash_S \Phi(\bar{m}, \bar{n})$;

and further,

(iv) $\vdash_S (E_1 x_2)\Phi(x_1, x_2)$.

This result not only yields that of Ehrenfeucht and Feferman as an immediate corollary but also gives a neat characterization of the class of partial recursive functions, in addition to showing that the condition (iii) of [**4**, p. 45] for definability of a total function by a formula Φ (viz. for each n

$$\vdash_S \Phi(\bar{n}, x_2) \wedge \Phi(\bar{n}, x_3) \supset x_2 = x_3)$$

implies the stronger condition obtained by replacing $\bar{n}$ by a variable

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x_1. The argument uses a theorem on exact separation of disjoint recursively enumerable sets due to Putnam and Smullyan; it is interesting to note that Shepherdson [3] obtains this separation result with his direct methods.

The present note gives a direct proof of a slight generalization of the theorem of Ritchie and Young alluded to above. Namely, let S be any consistent recursively enumerable theory in which every recursive *relation* is definable (in the sense of Tarski-Mostowski-Robinson [4, p. 44]) and which has a formula $x_1 \leqq x_2$ satisfying (i) above as well as (ii)′ below:

(ii)′ $\vdash_S x_1 \leqq x_2 \vee x_2 \leqq x_1$.

Alternatively, we may assume that S satisfies just (i) and that every recursive *function* is definable in S (cf. [3, p. 121]). In either case we have the following.

THEOREM. *Every partial recursive function is strongly representable in S.*

That is, if ϕ is a partial recursive function then there is a formula $\Phi(x_1, x_2)$ of S such that for all m, n, (iii) above holds as well as (iv).

2. **Weak representability.** We say that the partial function ϕ is *weakly represented* in S by $\Phi(x_1, x_2)$ provided that (iii) above holds as well as

(iva) $\vdash_S \Phi(x_1, x_2) \wedge \Phi(x_1, x_3) \supset x_2 = x_3$.

THEOREM 1. *Every partial recursive function ϕ is weakly representable in S.*

In what follows let ϕ be a *fixed* partial recursive function. Then according to the Enumeration Theorem of Kleene there is a recursive predicate $T(u, w)$ and a recursive function U such that

$$\phi(u) = v \Leftrightarrow (\exists w)[T(u, w) \ \& \ U(w) = v].$$

(We may take $T(u, w)$ as the $T_1(e, u, w)$ of [**IM**, p. 330], and the U found there, where e is any index for ϕ.) We may assume that in particular T has the property that $T(u, w_1) \ \& \ T(u, w_2) \Rightarrow w_1 = w_2$. Let the recursive relations $T(u, w)$ and $U(w) = v$ be defined in S by $\mathfrak{T}(x_1, x_3)$ and $\mathfrak{U}(x_3, x_2)$, respectively, and let $\mathfrak{D}(x, y, z)$ be the conjunction of the two formulas

$$\mathfrak{T}(x, z) \wedge (z_1)[z_1 \leqq z \supset [\mathfrak{T}(x, z_1) \supset z_1 = z]]$$

and

$$\mathfrak{U}(z, y) \wedge (y_1)[y_1 \leqq y \supset [\mathfrak{U}(z, y_1) \supset y_1 = y]].$$

LEMMA 1. *For any* u, v, w,

$$T(u, w) \ \& \ U(w) = v \Leftrightarrow \vdash_S \mathfrak{D}(\bar{u}, \bar{v}, \bar{w}).$$

PROOF. It suffices to show the implication to the right, so suppose that $T(u, w)$ and $U(w)=v$. We have then $\vdash_S \mathfrak{J}(\bar{u}, \bar{w}) \wedge \mathfrak{U}(\bar{w}, \bar{v})$ by the choice of $\mathfrak{J}$ and $\mathfrak{U}$. But also for every $n \leqq w$ we have

$$\vdash_S [\mathfrak{J}(\bar{u}, \bar{n}) \supset \bar{n} = \bar{w}]$$

(since the propositional components of this, for each fixed n, can be proved or disproved appropriately), so by (i) we have

$$\vdash_S (z_1)[z_1 \leqq \bar{w} \supset [\mathfrak{J}(\bar{u}, z_1) \supset z_1 = \bar{w}]].$$

In the same way we have

$$\vdash_S (y_1)[y_1 \leqq \bar{v} \supset [\mathfrak{U}(\bar{w}, y_1) \supset y_1 = \bar{v}]]$$

and thus $\vdash_S \mathfrak{D}(\bar{u}, \bar{v}, \bar{w})$.

LEMMA 2. $\vdash_S \mathfrak{D}(x, y_2, z_2) \wedge \mathfrak{D}(x, y_3, z_3) \supset y_2 = y_3$.

PROOF. From (ii)′ we have $\vdash_S z_2 \leqq z_3 \vee z_3 \leqq z_2$. Thus by specializing the variable z_1 in $\mathfrak{D}(x, y_2, z_2)$ to z_3, and in $\mathfrak{D}(x, y_3, z_3)$ to z_2, we obtain $\vdash_S \mathfrak{D}(x, y_2, z_2) \wedge \mathfrak{D}(x, y_3, z_3) \supset z_2 = z_3$. From this equality and by a similar manipulation of the final clauses of $\mathfrak{D}(x, y_2, z_2)$ and $\mathfrak{D}(x, y_3, z_3)$ we obtain the desired result.

PROOF OF THEOREM 1. Let Q, P be recursive predicates defined as follows:

$Q(u, v, w, q) \Leftrightarrow q$ is (the Gödel number of) a proof (in S) of $\mathfrak{D}(\bar{u}, \bar{v}, \bar{w})$

$P(u, v, r, p) \Leftrightarrow p$ is (the Gödel number of) a proof (in S) of $A_r^{(3)}(\bar{u}, \bar{v}, \bar{r})$

where $A_r^{(3)}(x_1, x_2, x_3)$ is the formula of S whose Gödel number is r and which contains no variables free other than x_1, x_2, x_3. Then P, Q are represented in S by some $\mathcal{P}$, $\mathcal{Q}$, respectively. Let r_0 be the Gödel number of the formula

$$(Ez)\,\mathfrak{D}(x_1, x_2, z) \wedge (x_4)[\mathcal{P}(x_1, x_2, x_3, x_4) \\ \supset (Ex_5)[x_5 \leqq x_4 \wedge (Ex_6)[x_6 \leqq x_5 \wedge \mathcal{Q}(x_1, x_2, x_6, x_5)]]].$$

We claim that ϕ is weakly represented by Φ, where $\Phi(x_1, x_2)$ is the formula $A_{r_0}^{(3)}(x_1, x_2, \bar{r}_0)$.

Now suppose that $\vdash_S \Phi(\bar{u}, \bar{v})$ for some u, v. Let p be (the number of) a proof; then $\vdash_S \mathcal{P}(\bar{u}, \bar{v}, \bar{r}_0, \bar{p})$ according to the definition of P by $\mathcal{P}$ and our supposition. Hence by specialization of x_4 to $\bar{p}$ in the definition of Φ we have (upon application of modus ponens) that

$$\vdash_S (Ex_5)[x_5 \leqq \bar{p} \wedge (Ex_6)[x_6 \leqq x_5 \wedge \mathfrak{Q}(\bar{u}, \bar{v}, x_6, x_5)]].$$

Hence (by a similar argument to the one above in connection with $\mathfrak{D}$), $Q(u, v, w, q)$ for some w, q with $w \leqq q \leqq p$, since Q is defined by $\mathfrak{Q}$. Hence (by the meaning of Q) there is a proof (in fact, with number q) of $\mathfrak{D}(\bar{u}, \bar{v}, \bar{w})$. We conclude both $T(u, w)$ and $U(w) = v$, and then that $\phi(u) = v$.

Conversely, let $\phi(u) = v$; take w minimal so that $T(u, w)$ and $U(w) = v$. Then for some minimal $q \geqq w$, q is a proof of $\mathfrak{D}(\bar{u}, \bar{v}, \bar{w})$ by Lemma 1. Note in passing that $\vdash_S (Ez)\mathfrak{D}(\bar{u}, \bar{v}, z)$ (by extension of the proof q). Now for $p < q$, we have $\vdash_S \sim \mathcal{P}(\bar{u}, \bar{v}, \bar{r}_0, \bar{p})$, since otherwise entails (as above) the existence of w_1, q_1, p_1 with $w_1 \leqq q_1 \leqq p_1 < q$ where q_1 is a proof of $\mathfrak{D}(\bar{u}, \bar{v}, \bar{w}_1)$; but then $T(u, w_1)$ implies $w_1 = w$, so $q \leqq q_1$ (contradicting the choice of q). Hence we can show by (i) and (ii) (or (ii)$'$) that $\vdash_S (x_4)[\mathcal{P}(\bar{u}, \bar{v}, \bar{r}_0, x_4) \supset \bar{q} \leqq x_4]$. But then

$$\vdash_S (x_4)[\mathcal{P}(\bar{u}, \bar{v}, \bar{r}_0, x_4) \supset \bar{q} \leqq x_4 \wedge \bar{w} \leqq \bar{q} \wedge \mathfrak{Q}(\bar{u}, \bar{v}, \bar{w}, \bar{q})],$$

so we conclude (by existential quantifications), with the help of the previously noted fact that $\vdash_S (Ez)\mathfrak{D}(\bar{u}, \bar{v}, z)$, that $\vdash_S \Phi(\bar{u}, \bar{v})$.

All that remains to be shown is (iva), but this follows immediately from Lemma 2.

3. **Strong representability.** Now we show how to construct another Rosser-type argument to obtain the strong representability of ϕ. To this end recall the definition of $A_r^{(3)}(x_1, x_2, x_3)$ given above. Define recursive M, N as follows:

$M(u, v, r, p) \Leftrightarrow p$ is (the number of) a proof (in S) of $\sim\Phi(\bar{u}, \bar{v}) \supset A_r^{(3)}(\bar{u}, \bar{v}, \bar{r})$,

$N(u, v, r, q) \Leftrightarrow q$ is a proof of $\sim\Phi(\bar{u}, \bar{v}) \supset \sim A_r^{(3)}(\bar{u}, \bar{v}, \bar{r})$.

Let M, N be defined in S by $\mathfrak{M}$, $\mathfrak{N}$, respectively, and let r_0 be the Gödel number of the formula

$$(x_4)[\mathfrak{M}(x_1, x_2, x_3, x_4) \supset (Ex_5)[x_3 \leqq x_2 \wedge \mathfrak{N}(x_1, x_2, x_3, x_5)]].$$

Take R_0 as $A_{r_0}^{(3)}(x_1, x_2, \bar{r}_0)$ and $\Phi^*(x_1, x_2)$ as the formula

$$\Phi(x_1, x_2) \vee [(z) \sim \Phi(x_1, z) \wedge [(x_2 = \bar{0} \wedge R_0(x_1, x_2)) \vee (x_2 = \bar{1} \wedge \sim R_0(x_1, x_2))]].$$

Theorem 2. *Φ^* strongly represents ϕ in S.*

Proof. By Theorem 1 we infer that if $\phi(u) = v$, then $\vdash_S \Phi(\bar{u}, \bar{v})$, so also $\vdash_S \Phi^*(\bar{u}, \bar{v})$. With the help of the same theorem $\vdash_S \Phi^*(x, y) \wedge \Phi^*(x, z) \supset y = z$. It is straightforward to verify that $\vdash_S (Ey)\Phi^*(x, y)$, merely using the logical form of Φ^*, so it remains to be shown that if

$\Phi^*(\bar{u}, \bar{v})$ is provable in S for some u, v, then in fact $\phi(u)=v$. Thus suppose that $\vdash_S\Phi^*(\bar{u}, \bar{v})$; it suffices to show that $\vdash_S\Phi(\bar{u}, \bar{v})$ also. First note that if $v>1$, then $\vdash_S \bar{v}\neq\bar{0}\wedge\bar{v}\neq\bar{1}$ by our assumptions about S. But for such v it easily follows from the form of $\Phi^*(\bar{u}, \bar{v})$ that $\vdash_S\Phi(\bar{u}, \bar{v})$. Hence we need consider only the cases $v=0$ and $v=1$.

Case $v=0$. In this case we see that $\vdash_S{\sim}\Phi(\bar{u}, \bar{0}) \supset R_0(\bar{u}, \bar{0})$; let p be the number of a proof. Note that $M(u, 0, r_0, p)$ holds then, so that $\vdash_S\mathfrak{M}(\bar{u}, \bar{0}, \bar{r}_0, \bar{p})$ since $\mathfrak{M}$ defines M. Now extend the proof p by specializing the x_4 in the definition of R_0 to $\bar{p}$, to obtain

$$(0^*) \qquad \vdash_S \sim \Phi(\bar{u}, \bar{0}) \supset (Ex_5)[x_5 \leqq \bar{p} \wedge \mathfrak{N}(\bar{u}, \bar{0}, \bar{r}_0, x_5)].$$

Now two possibilities arise. *First*, that $\vdash_S{\sim}\Phi(\bar{u}, \bar{0})\supset{\sim}R_0(\bar{u}, \bar{0})$. In this subcase, clearly $\vdash_S\Phi(\bar{u}, \bar{0})$ as desired. *Otherwise*, it is not the case that $\vdash_S{\sim}\Phi(\bar{u}, \bar{0})\supset{\sim}R_0(\bar{u}, \bar{0})$, so $N(u, 0, r_0, q)$ is false for all q, and in particular for $q\leqq p$. Hence for such q, we have $\vdash_S{\sim}\mathfrak{N}(\bar{u}, \bar{0}, \bar{r}_0, \bar{q})$, since $\mathfrak{N}$ defines N, and thus $\vdash_S(x_5)[x_5\leqq\bar{p}\supset{\sim}\mathfrak{N}(\bar{u}, \bar{0}, \bar{r}_0, x_5)]$. From this and (0*) we conclude that $\vdash_S\Phi(\bar{u}, \bar{0})$.

Case $v=1$. Now we see that $\vdash_S{\sim}\Phi(\bar{u}, \bar{1})\supset{\sim}R_0(\bar{u}, \bar{1})$; if q is the number of a proof, $N(u, 1, r_0, q)$ holds, so $\vdash_S\mathfrak{N}(\bar{u}, \bar{1}, \bar{r}_0, \bar{q})$. Again it is possible that $\vdash_S{\sim}\Phi(\bar{u}, \bar{1})\supset R_0(\bar{u}, \bar{1})$; and, if so, then $\vdash_S\Phi(\bar{u}, \bar{1})$ as desired. Otherwise, there is no proof in S of ${\sim}\Phi(\bar{u}, \bar{1})\supset R_0(\bar{u}, \bar{1})$, and so $M(u,1,r_0,p)$ fails for all p. In particular, for $p\leqq q$ we have $\vdash_S{\sim}\mathfrak{M}(\bar{u}, \bar{1}, \bar{r}_0, \bar{p})$, and so (by (i)) $\vdash_S\mathfrak{M}(\bar{u}, \bar{1}, \bar{r}_0, x_4)\supset{\sim}(x_4\leqq\bar{q})$. Now from a proof of $\mathfrak{N}(\bar{u}, \bar{1}, \bar{r}_0, \bar{q})$ we can construct one of $\bar{q}\leqq x_4\supset\bar{q}\leqq x_4\wedge\mathfrak{N}(\bar{u}, \bar{1}, \bar{r}_0, \bar{q})$, so $\vdash_S\bar{q}\leqq x_4\supset(Ex_5)[x_5\leqq x_4\wedge\mathfrak{N}(\bar{u}, \bar{1}, \bar{r}_0, x_5)]$. By (ii) we have $\vdash_S x_4\leqq\bar{q}\vee\bar{q}\leqq x_4$; so by combining this with the above results,

$$(1^*) \qquad \vdash_S\mathfrak{M}(\bar{u}, \bar{1}, \bar{r}_0 x_4) \supset (Ex_5)[x_5 \leqq x_4 \wedge \mathfrak{N}(\bar{u}, \bar{1}, \bar{r}_0, x_5)].$$

Generalize on x_4 in (1*) to obtain $\vdash_S R_0(\bar{u}, \bar{1})$, and conclude finally that $\vdash_S\Phi(\bar{u}, \bar{1})$ in this case also.

Bibliography

1. A. Ehrenfeucht and S. Feferman, *Representability of recursively enumerable sets in formal theories*, Arch. Math. Logik Grundlagenforsch. **5** (1960), 37–41.

2. R. W. Ritchie and P. R. Young, *Strong representability of partial recursive functions*, Notices Amer. Math. Soc. **13** (1966), 114.

3. J. C. Shepherdson, *Representability of recursively enumerable sets in formal theories*, Arch. Math. Logik Grundlagenforsch. **5** (1960), 119–127.

4. A. Tarski, A. Mostowski and R. M. Robinson, *Undecidable theories*, North-Holland, Amsterdam, 1953.

IM. S. C. Kleene, *Introduction to metamathematics*, Van Nostrand, New York, 1952.

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