1. Introduction. Ehrenfeucht and Feferman have shown [1] that all recursively enumerable sets \( X \) of natural numbers are "representable" in any consistent recursively enumerable theory \( S \) in which all recursive functions are definable (in the sense of Tarski-Mostowski-Robinson [4]) and which has a formula \( x_1 \leq x_2 \) satisfying conditions (i), (ii) below for each natural number \( n \):

(i) \( \vdash_S x_1 \leq n \equiv x_1 = 0 \lor x_1 = 1 \lor \ldots \lor x_1 = n \),

(ii) \( \vdash_S x_1 \leq n \lor \bar{n} \equiv x_1 = n \).

(Here \( \bar{n} \) is the (closed) numerical term of \( S \) corresponding to \( n \), i.e. \( \Delta_n \) of [4, p. 44].) (By a construction of Cobham (see [3, p. 121] for details), (ii) is redundant in the presence of (i) and the definability in \( S \) of the successor function.) That is, for such an \( X \), there is a formula \( \Phi(x_1) \) of \( S \) (with exactly one free variable \( x_1 \)) such that for every \( n \),

\[
\neg \Phi(n) \iff n \in X.
\]

The argument is to show that there is some creative set \( C \) representable in \( S \), from which the result follows by the reducibility of \( X \) to \( C \) by some recursive function (Myhill). Shepherdson has obtained the result [3] more directly by an elegant adaptation of Rosser-type arguments, much as Bernays obtained results of Myhill on theories. In [2] Ritchie and Young show that in every consistent recursively enumerable extension \( S \) of R. M. Robinson's system \( R \) ([4, pp. 52–53]), all partial recursive functions \( \phi \) are "strongly representable." That is, for such a \( \phi \), there is a formula \( \Phi(x_1, x_2) \) of \( S \) such that for all \( m, n \),

(iii) \( \phi(m) = n \iff \vdash_S \Phi(m, n) \);

and further,

(iv) \( \vdash_S (E_{x_2} \Phi(x_1, x_2)) \).

This result not only yields that of Ehrenfeucht and Feferman as an immediate corollary but also gives a neat characterization of the class of partial recursive functions, in addition to showing that the condition (iii) of [4, p. 45] for definability of a total function by a formula \( \Phi \) (viz. for each \( n \))

\[
\vdash_S \Phi(n, x_2) \land \Phi(n, x_3) \supset x_2 = x_3
\]

implies the stronger condition obtained by replacing \( n \) by a variable.

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The argument uses a theorem on exact separation of disjoint recursively enumerable sets due to Putnam and Smullyan; it is interesting to note that Shepherdson [3] obtains this separation result with his direct methods.

The present note gives a direct proof of a slight generalization of the theorem of Ritchie and Young alluded to above. Namely, let \( S \) be any consistent recursively enumerable theory in which every recursive relation is definable (in the sense of Tarski-Mostowski-Robinson [4, p. 44]) and which has a formula \( x_1 \leq x_2 \) satisfying (i) above as well as (ii)' below:

\[(ii)\' \quad \vdash_S x_1 \leq x_2 \lor x_2 \leq x_1.\]

Alternatively, we may assume that \( S \) satisfies just (i) and that every recursive function is definable in \( S \) (cf. [3, p. 121]). In either case we have the following.

**Theorem.** Every partial recursive function is strongly representable in \( S \).

That is, if \( \phi \) is a partial recursive function then there is a formula \( \Phi(x_1, x_2) \) of \( S \) such that for all \( m, n \), (iii) above holds as well as (iv).

2. Weak representability. We say that the partial function \( \phi \) is weakly represented in \( S \) by \( \Phi(x_1, x_2) \) provided that (iii) above holds as well as

\[(iva) \quad \vdash_S \Phi(x_1, x_2) \land \Phi(x_1, x_3) \supseteq x_2 = x_3.\]

**Theorem 1.** Every partial recursive function \( \phi \) is weakly representable in \( S \).

In what follows let \( \phi \) be a fixed partial recursive function. Then according to the Enumeration Theorem of Kleene there is a recursive predicate \( T(u, w) \) and a recursive function \( U \) such that

\[\phi(u) = v \iff (\exists w)[T(u, w) \land U(w) = v].\]

(We may take \( T(u, w) \) as the \( T_1(e, u, w) \) of [IM, p. 330], and the \( U \) found there, where \( e \) is any index for \( \phi \).) We may assume that in particular \( T \) has the property that \( T(u, w_1) \land T(u, w_2) \Rightarrow w_1 = w_2 \). Let the recursive relations \( T(u, w) \) and \( U(w) = v \) be defined in \( S \) by \( \exists(x_1, x_2) \) and \( \forall(x_3, x_2) \), respectively, and let \( \exists(x, y, z) \) be the conjunction of the two formulas

\[\exists(x, z) \land (s_1)[s_1 \leq z \supset [\exists(x, s_1) \supset z_1 = z]]\]

and

\[\forall(z, y) \land (y_1)[y_1 \leq y \supset [\forall(z, y_1) \supset y_1 = y]].\]
**Lemma 1.** For any $u, v, w,$

$$T(u, w) \& U(w) = v \iff \vdash_S \mathcal{D}(\bar{u}, \bar{v}, \bar{w}).$$

**Proof.** It suffices to show the implication to the right, so suppose that $T(u, w)$ and $U(w) = v.$ We have then $\vdash_S \exists \bar{u}(\bar{u}, \bar{w}) \& \forall \bar{w}(\bar{v})$ by the choice of $\exists$ and $\forall.$ But also for every $n \leq w$ we have

$$\vdash_S [\exists(\bar{u}, \bar{v}) \supset \bar{u} = \bar{w}]$$

(since the propositional components of this, for each fixed $n,$ can be proved or disproved appropriately), so by (i) we have

$$\vdash_S (z_1) [z_1 \leq \bar{w} \supset [\exists(\bar{u}, \bar{v}) \supset z_1 = \bar{w}]].$$

In the same way we have

$$\vdash_S (y_1) [y_1 \leq \bar{v} \supset [\exists(\bar{w}, \bar{v}) \supset y_1 = \bar{v}]]$$

and thus $\vdash_S \mathcal{D}(\bar{u}, \bar{v}, \bar{w}).$

**Lemma 2.** $\vdash_S \mathcal{D}(x, y_2, z_2) \& \mathcal{D}(x, y_3, z_3) \supset y_2 = y_3.$

**Proof.** From (ii)' we have $\vdash_S z_2 \leq z_3 \lor z_3 \leq z_2.$ Thus by specializing the variable $z_1$ in $\mathcal{D}(x, y_2, z_2)$ to $z_3,$ and in $\mathcal{D}(x, y_3, z_3)$ to $z_2,$ we obtain $\vdash_S \mathcal{D}(x, y_2, z_2) \& \mathcal{D}(x, y_3, z_3) \supset z_2 = z_3.$ From this equality and by a similar manipulation of the final clauses of $\mathcal{D}(x, y_2, z_2)$ and $\mathcal{D}(x, y_3, z_3)$ we obtain the desired result.

**Proof of Theorem 1.** Let $Q, P$ be recursive predicates defined as follows:

$Q(u, v, w, q) \iff q$ is (the Gödel number of) a proof (in $\mathcal{S}$) of $\mathcal{D}(\bar{u}, \bar{v}, \bar{w})$

$P(u, v, r, p) \iff p$ is (the Gödel number of) a proof (in $\mathcal{S}$) of $A^3_r(\bar{u}, \bar{v}, \bar{r})$

where $A^3_r(x_1, x_2, x_3)$ is the formula of $\mathcal{S}$ whose Gödel number is $r$ and which contains no variables free other than $x_1, x_2, x_3.$ Then $P, Q$ are represented in $\mathcal{S}$ by some $\mathcal{P}, \mathcal{Q},$ respectively. Let $r_0$ be the Gödel number of the formula

$$(Ez) \mathcal{D}(x_1, x_2, z) \& (x_4) \mathcal{[\mathcal{P}(x_1, x_2, x_3, x_4) \supset (Ex_6)[x_6 \leq x_4 \& (Ex_6)[x_6 \leq x_5 \& Q(x_1, x_2, x_6, x_6)]].$$

We claim that $\phi$ is weakly represented by $\phi,$ where $\Phi(x_1, x_2)$ is the formula $A^3(\bar{x}_1, \bar{x}_2, \bar{r}_0).$

Now suppose that $\vdash_S \Phi(\bar{u}, \bar{v})$ for some $u, v.$ Let $p$ be (the number of) a proof; then $\vdash_s \Phi(\bar{u}, \bar{v}, \bar{r}_0, \bar{p})$ according to the definition of $P$ by $\Phi$ and our supposition. Hence by specialization of $x_4$ to $\bar{p}$ in the definition of $\Phi$ we have (upon application of modus ponens) that
Hence (by a similar argument to the one above in connection with \( \mathcal{D} \)), \( Q(u, v, w, q) \) for some \( w, q \) with \( w \leq q \leq p \), since \( Q \) is defined by \( \mathcal{Q} \). Hence (by the meaning of \( Q \)) there is a proof (in fact, with number \( q \)) of \( \mathcal{D}(\bar{u}, \bar{v}, \bar{w}) \). We conclude both \( T(u, w) \) and \( U(w) = v \), and then that \( \phi(u) = v \).

Conversely, let \( \phi(u) = v \); take \( w \) minimal so that \( T(u, w) \) and \( U(w) = v \). Then for some minimal \( q \geq w \), \( q \) is a proof of \( \mathcal{D}(\bar{u}, \bar{v}, \bar{w}) \) by Lemma 1. Note in passing that \(-s(Ez)\mathcal{D}(u, v, z)\) (by extension of the proof \( q \)). Now for \( p < q \), we have \(-s(\mathcal{D}(u, v, f_0, f))\), since otherwise entails (as above) the existence of \( w_1, q_1, p_1 \) with \( w_1 \leq q_1 \leq p_1 < q \) where \( q_1 \) is a proof of \( \mathcal{D}(\bar{u}, \bar{v}, \bar{w_1}) \); but then \( T(u, w_1) \) implies \( w_1 = w \), so \( q \leq q_1 \) (contradicting the choice of \( q \)). Hence we can show by (i) and (ii) (or (ii)') that \(-s(\mathcal{D}(u, v, f_0, x_4) \supset q \leq x_4)\). But then

\[
-s(\mathcal{D}(u, v, r_0, x_4) \supset q \leq x_4 \land \bar{w} \leq \bar{q} \land \mathcal{Q}(\bar{u}, \bar{v}, \bar{w}, \bar{q})],
\]

so we conclude (by existential quantifications), with the help of the previously noted fact that \(-s(Ez)\mathcal{D}(u, v, z)\), that \(-s\phi(\bar{u}, \bar{v})\).

All that remains to be shown is (iva), but this follows immediately from Lemma 2.

3. Strong representability. Now we show how to construct another Rosser-type argument to obtain the strong representability of \( \phi \). To this end recall the definition of \( A^{(3)}_r(x_1, x_2, x_3) \) given above. Define recursive \( M, N \) as follows:

\[
M(u, v, r, p) \iff p \text{ is (the number of) a proof (in } S \text{) of } \sim \phi(\bar{u}, \bar{v}) \supset A^{(3)}_r(\bar{u}, \bar{v}, \bar{r}),
\]

\[
N(u, v, r, q) \iff q \text{ is a proof of } \sim \phi(\bar{u}, \bar{v}) \supset A^{(3)}_r(\bar{u}, \bar{v}, \bar{r}).
\]

Let \( M, N \) be defined in \( S \) by \( \mathfrak{M}, \mathfrak{P} \), respectively, and let \( r_0 \) be the Gödel number of the formula

\[
(x_4)[\mathfrak{M}(x_1, x_2, x_3, x_4) \supset (Ex_6)[x_3 \leq x_2 \land \mathfrak{P}(x_1, x_2, x_3, x_6)]].
\]

Take \( R_0 \) as \( A^{(3)}_r(x_1, x_2, r_0) \) and \( \Phi^*(x_1, x_2) \) as the formula

\[
\Phi(x_1, x_2) \lor [(z) \sim \Phi(x_1, z) \land [(x_2 = \bar{0} \land R_0(x_1, x_2)) \lor (x_2 = \bar{1} \land \sim R_0(x_1, x_2))].
\]

**Theorem 2.** \( \Phi^* \) strongly represents \( \phi \) in \( S \).

**Proof.** By Theorem 1 we infer that if \( \phi(u) = v \), then \(-s\phi(\bar{u}, \bar{v})\), so also \(-s\Phi^*(x, y) \land \Phi^*(x, z) \supset y = z \). It is straightforward to verify that \(-s(Ex)\Phi^*(x, y)\), merely using the logical form of \( \Phi^* \), so it remains to be shown that if


\( \Phi^*(\bar{u}, \bar{v}) \) is provable in \( S \) for some \( u, v \), then in fact \( \phi(u) = v \). Thus suppose that \( \vdash_S \Phi^*(\bar{u}, \bar{v}) \); it suffices to show that \( \vdash_S \Phi(\bar{u}, \bar{v}) \) also. First note that if \( v > 1 \), then \( \vdash_S \bar{v} \neq \bar{0} \land \bar{v} \neq \bar{1} \) by our assumptions about \( S \). But for such \( v \) it easily follows from the form of \( \Phi^*(\bar{u}, \bar{v}) \) that \( \vdash_S \Phi(\bar{u}, \bar{v}) \). Hence we need consider only the cases \( v = 0 \) and \( v = 1 \).

**Case** \( v = 0 \). In this case we see that \( \vdash_S \sim \Phi(\bar{u}, \bar{0}) \supset R_0(\bar{u}, \bar{0}) \); let \( \bar{p} \) be the number of a proof. Note that \( M(u, 0, r_0, \bar{p}) \) holds then, so that \( \vdash_S \exists \alpha(\bar{u}, \bar{0}, r_0, \bar{p}) \) since \( \exists \alpha \) defines \( M \). Now extend the proof \( \bar{p} \) by specializing the \( x_4 \) in the definition of \( R_0 \) to \( \bar{p} \), to obtain

\[
\vdash_S \sim \Phi(\bar{u}, \bar{0}) \supset (Ex_{\bar{5}})[x_5 \leq \bar{p} \land \alpha(\bar{u}, \bar{0}, r_0, x_5)].
\]

Now two possibilities arise. First, that \( \vdash_S \sim \Phi(\bar{u}, \bar{0}) \supset \sim R_0(\bar{u}, \bar{0}) \). In this subcase, clearly \( \vdash_S \Phi(\bar{u}, \bar{0}) \) as desired. **Otherwise**, it is not the case that \( \vdash_S \sim \Phi(\bar{u}, \bar{0}) \supset \sim R_0(\bar{u}, \bar{0}) \), so \( \neg N(u, 0, r_0, q) \) is false for all \( q \), and in particular for \( q \leq \bar{p} \). Hence for such \( q \), we have \( \vdash_S \sim \exists \alpha(\bar{u}, \bar{0}, r_0, \bar{q}) \), since \( \exists \alpha \) defines \( N \), and thus \( \vdash_S (x_{\bar{5}})[x_5 \leq \bar{p} \supset \exists \alpha(\bar{u}, \bar{0}, r_0, x_{\bar{5}})] \). From this and (0*) we conclude that \( \vdash_S \Phi(\bar{u}, \bar{0}) \).

**Case** \( v = 1 \). Now we see that \( \vdash_S \sim \Phi(\bar{u}, \bar{1}) \supset \sim R_0(\bar{u}, \bar{1}) \); if \( q \) is the number of a proof, \( N(u, 1, r_0, q) \) holds, so \( \vdash_S \exists \alpha(\bar{u}, \bar{1}, r_0, \bar{q}) \). Again it is possible that \( \vdash_S \sim \Phi(\bar{u}, \bar{1}) \supset R_0(\bar{u}, \bar{1}) \); and, if so, then \( \vdash_S \Phi(\bar{u}, \bar{1}) \) as desired. **Otherwise**, there is no proof in \( S \) of \( \sim \Phi(\bar{u}, \bar{1}) \supset R_0(\bar{u}, \bar{1}) \), and so \( M(u, 1, r_0, \bar{p}) \) fails for all \( \bar{p} \). In particular, for \( \bar{p} \leq q \) we have \( \vdash_S \exists \alpha(\bar{u}, \bar{1}, r_0, \bar{p}) \), and so (by (i)) \( \vdash_S \exists \alpha(\bar{u}, \bar{1}, r_0, x_{\bar{4}}) \supset (x_{\bar{4}} \leq \bar{q}) \).

Now from a proof of \( \alpha(\bar{u}, \bar{1}, r_0, \bar{q}) \) we can construct one of \( \bar{q} \leq x_{\bar{4}} \supset \bar{q} \leq x_{\bar{4}} \supset \alpha(\bar{u}, \bar{1}, r_0, \bar{q}) \), so \( \vdash_S \exists \bar{q} \supset (Ex_{\bar{5}})[x_5 \leq x_{\bar{4}} \land \exists \alpha(\bar{u}, \bar{1}, \bar{r}_0, x_{\bar{5}})] \). By (ii) we have \( \vdash_S x_{\bar{4}} \leq \bar{q} \land \bar{q} \leq x_{\bar{4}} \); so by combining this with the above results,

\[
\vdash_S \exists \alpha(\bar{u}, \bar{1}, r_0, x_{\bar{4}}) \supset (Ex_{\bar{5}})[x_5 \leq x_{\bar{4}} \land \alpha(\bar{u}, \bar{1}, r_0, x_{\bar{5}})].
\]

Generalize on \( x_4 \) in (1*) to obtain \( \vdash_S R_0(\bar{u}, \bar{1}) \), and conclude finally that \( \vdash_S \Phi(\bar{u}, \bar{1}) \) in this case also.

**Bibliography**


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