1. Introduction. Ehrenfeucht and Feferman have shown [1] that all recursively enumerable sets \( X \) of natural numbers are “representable” in any consistent recursively enumerable theory \( S \) in which all recursive functions are definable (in the sense of Tarski-Mostowski-Robinson [4]) and which has a formula \( x_1 \leq x_2 \) satisfying conditions (i), (ii) below for each natural number \( n \):

(i) \( \vdash_S x_1 \leq \overline{n} \Rightarrow x_1 = 0 \lor x_1 = 1 \lor \cdots \lor x_1 = \overline{n} \),

(ii) \( \vdash_S x_1 \leq \overline{n} \lor \overline{n} \leq x_1 \).

(Here \( \overline{n} \) is the (closed) numerical term of \( S \) corresponding to \( n \), i.e. \( \Delta_n \) of [4, p. 44].) (By a construction of Cobham (see [3, p. 121] for details), (ii) is redundant in the presence of (i) and the definability in \( S \) of the successor function.) That is, for such an \( X \), there is a formula \( \Phi(x_1) \) of \( S \) (with exactly one free variable \( x_1 \)) such that for every \( n \),

\[ n \in X \iff \vdash_S \Phi(\overline{n}). \]

The argument is to show that there is some creative set \( C \) representable in \( S \), from which the result follows by the reducibility of \( X \) to \( C \) by some recursive function (Myhill). Shepherdson has obtained the result [3] more directly by an elegant adaptation of Rosser-type arguments, much as Bernays obtained results of Myhill on theories. In [2] Ritchie and Young show that in every consistent recursively enumerable extension \( S \) of R. M. Robinson’s system \( R \) ([4, pp. 52–53]), all partial recursive functions \( \phi \) are “strongly representable.” That is, for such a \( \phi \), there is a formula \( \Phi(x_1, x_2) \) of \( S \) such that for all \( m, n \),

(iii) \( \phi(m) = n \iff \vdash_S \Phi(\overline{m}, \overline{n}) \);

and further,

(iv) \( \vdash_S (E_1 x_2) \Phi(x_1, x_2) \).

This result not only yields that of Ehrenfeucht and Feferman as an immediate corollary but also gives a neat characterization of the class of partial recursive functions, in addition to showing that the condition (iii) of [4, p. 45] for definability of a total function by a formula \( \Phi \) (viz. for each \( n \)

\[ \vdash_S \Phi(\overline{n}, x_2) \land \Phi(\overline{n}, x_3) \supset x_2 = x_3 \]

implies the stronger condition obtained by replacing \( \overline{n} \) by a variable...
The argument uses a theorem on exact separation of disjoint recursively enumerable sets due to Putnam and Smullyan; it is interesting to note that Shepherdson [3] obtains this separation result with his direct methods.

The present note gives a direct proof of a slight generalization of the theorem of Ritchie and Young alluded to above. Namely, let $S$ be any consistent recursively enumerable theory in which every recursive relation is definable (in the sense of Tarski-Mostowski-Robinson [4, p. 44]) and which has a formula $x_1 \leq x_2$ satisfying (i) above as well as (ii)' below:

$$(ii)' \vdash_S x_1 \leq x_2 \lor x_2 \leq x_1.$$  
Alternatively, we may assume that $S$ satisfies just (i) and that every recursive function is definable in $S$ (cf. [3, p. 121]). In either case we have the following.

**Theorem.** Every partial recursive function is strongly representable in $S$.

That is, if $\phi$ is a partial recursive function then there is a formula $\Phi(x_1, x_2)$ of $S$ such that for all $m, n$, (iii) above holds as well as (iv).

**2. Weak representability.** We say that the partial function $\phi$ is weakly represented in $S$ by $\Phi(x_1, x_2)$ provided that (iii) above holds as well as

$$(iva) \quad \vdash_S \Phi(x_1, x_2) \land \Phi(x_1, x_3) \supset x_2 = x_3.$$  

**Theorem 1.** Every partial recursive function $\phi$ is weakly representable in $S$.

In what follows let $\phi$ be a fixed partial recursive function. Then according to the Enumeration Theorem of Kleene there is a recursive predicate $T(u, w)$ and a recursive function $U$ such that

$$\phi(u) = v \iff (\exists w)[T(u, w) \land U(w) = v].$$  
(We may take $T(u, w)$ as the $T_1(e, u, w)$ of [IM, p. 330], and the $U$ found there, where $e$ is any index for $\phi$.) We may assume that in particular $T$ has the property that $T(u, w_1) \land T(u, w_2) \implies w_1 = w_2$. Let the recursive relations $T(u, w)$ and $U(w) = v$ be defined in $S$ by $\exists(x_1, x_2)$ and $U(x_3, x_2)$, respectively, and let $\forall(x, y, z)$ be the conjunction of the two formulas

$$\exists(x, z) \land (z_1)[z_1 \leq z \supset [\exists(x, z_1) \supset z_1 = z]]$$

and

$$\forall(z, y) \land (y_1)[y_1 \leq y \supset [\forall(z, y_1) \supset y_1 = y]].$$
Lemma 1. For any $u, v, w$,

$$T(u, w) \land U(w) = v \iff \neg s \mathcal{D}(\bar{u}, \bar{v}, \bar{w}).$$

Proof. It suffices to show the implication to the right, so suppose that $T(u, w)$ and $U(w) = v$. We have then $\neg s \mathcal{J}(\bar{u}, \bar{w}) \land \mathcal{U}(\bar{w}, \bar{v})$ by the choice of $\mathcal{J}$ and $\mathcal{U}$. But also for every $n \leq w$ we have

$$\neg s [\mathcal{J}(\bar{u}, \bar{n}) \supset \bar{n} = \bar{w}]$$

(since the propositional components of this, for each fixed $n$, can be proved or disproved appropriately), so by (i) we have

$$\neg s (z_1) [z_1 \leq \bar{w} \supset [\mathcal{J}(\bar{u}, z_1) \supset z_1 = \bar{w}]].$$

In the same way we have

$$\neg s (y_1) [y_1 \leq \bar{v} \supset [\mathcal{U}(\bar{w}, y_1) \supset y_1 = \bar{v}]],$$

and thus $\neg s \mathcal{D}(\bar{u}, \bar{v}, \bar{w})$.

Lemma 2. $\neg s \mathcal{D}(x, y_2, z_2) \land \mathcal{D}(x, y_3, z_3) \supset y_2 = y_3$.

Proof. From (ii)' we have $\neg s z_2 \leq z_3 \lor z_3 \leq z_2$. Thus by specializing the variable $z_1$ in $\mathcal{D}(x, y_2, z_2)$ to $z_3$, and in $\mathcal{D}(x, y_3, z_3)$ to $z_2$, we obtain $\neg s \mathcal{D}(x, y_2, z_2) \land \mathcal{D}(x, y_3, z_3) \supset z_2 = z_3$. From this equality and by a similar manipulation of the final clauses of $\mathcal{D}(x, y_2, z_2)$ and $\mathcal{D}(x, y_3, z_3)$ we obtain the desired result.

Proof of Theorem 1. Let $Q, P$ be recursive predicates defined as follows:

$Q(u, v, w, q) \iff q$ is (the G"{o}del number of) a proof (in $S$) of $\mathcal{D}(\bar{u}, \bar{v}, \bar{w})$

$P(u, v, r, p) \iff p$ is (the G"{o}del number of) a proof (in $S$) of $A_r^{(3)}(\bar{u}, \bar{v}, \bar{r})$

where $A_r^{(3)}(x_1, x_2, x_3)$ is the formula of $S$ whose G"{o}del number is $r$ and which contains no variables free other than $x_1, x_2, x_3$. Then $P, Q$ are represented in $S$ by some $\Phi, \Psi$, respectively. Let $r_0$ be the G"{o}del number of the formula

$$(Ez) \mathcal{D}(x_1, x_2, z) \land (x_4)[\Phi(x_1, x_2, x_3, x_4)$$

$$\supset (Ez_6)[x_6 \leq x_4 \land (Ez_6)[x_6 \leq x_5 \land Q(x_1, x_2, x_6, x_6)]]].$$

We claim that $\phi$ is weakly represented by $\Phi$, where $\Phi(x_1, x_2)$ is the formula $A_r^{(3)}(x_1, x_2, \bar{r}_0)$.

Now suppose that $\neg s \mathcal{D}(\bar{u}, \bar{v})$ for some $u, v$. Let $p$ be (the number of) a proof; then $\neg s \mathcal{D}(\bar{u}, \bar{v}, \bar{r}_0, \bar{p})$ according to the definition of $P$ by $\Phi$ and our supposition. Hence by specialization of $x_4$ to $\bar{p}$ in the definition of $\Phi$ we have (upon application of modus ponens) that
Hence (by a similar argument to the one above in connection with $\mathcal{D}$), $Q(u, v, w, q)$ for some $w, q$ with $w \leq q \leq p$, since $Q$ is defined by $y$. Hence (by the meaning of $Q$) there is a proof (in fact, with number $q$) of $\mathcal{D}(u, \bar{v}, \bar{w})$. We conclude both $T(u, w)$ and $U(w) = v$, and then that $\phi(u) = v$.

Conversely, let $\phi(u) = v$; take $w$ minimal so that $T(u, w)$ and $U(w) = v$. Then for some minimal $q \geq w$, $q$ is a proof of $\mathcal{D}(u, \bar{v}, \bar{w})$ by Lemma 1. Note in passing that $\neg \psi(Ez) \mathcal{D}(u, v, z)$ (by extension of the proof $q$). Now for $p < q$, we have $\neg \psi(\mathcal{D}(u, \bar{v}, \bar{w})$, since otherwise entails (as above) the existence of $w_1, q_1, p_1$ with $w_1 \leq q_1 \leq p_1 < q$ where $q_1$ is a proof of $\mathcal{D}(u, \bar{v}, w_1)$; but then $T(u, w_1)$ implies $w_1 = w$, so $q \leq q_1$ (contradicting the choice of $q$). Hence we can show by (i) and (ii) or (ii)' that $\neg \psi(x_4)[\mathcal{D}(u, \bar{v}, \bar{r}_0, x_4) \supset \bar{q} \leq x_4 \land \bar{w} \leq \bar{q} \land \mathcal{Q}(u, \bar{v}, \bar{w}, \bar{q})]$, so we conclude (by existential quantifications), with the help of the previously noted fact that $\neg \psi(Ez) \mathcal{D}(u, v, z)$, that $\neg \psi(u, v)$.

All that remains to be shown is (iva), but this follows immediately from Lemma 2.

3. **Strong representability.** Now we show how to construct another Rosser-type argument to obtain the strong representability of $\phi$. To this end recall the definition of $A^{(3)}_r(x_1, x_2, x_3)$ given above. Define recursive $M, N$ as follows:

$M(u, v, r, p) \equiv p$ is (the number of) a proof (in $S$) of $\neg \Phi(u, \bar{v})$ $\supset A^{(3)}_r(u, \bar{v}, \bar{r})$,

$N(u, v, r, q) \equiv q$ is a proof of $\neg \Phi(u, \bar{v})$ $\supset A^{(3)}_r(u, \bar{v}, \bar{r})$.

Let $M, N$ be defined in $S$ by $\mathfrak{M}, \mathfrak{N}$, respectively, and let $r_0$ be the Gödel number of the formula

$$(x_4)[\mathfrak{M}(x_1, x_2, x_3, x_4) \supset (Ex_6)[x_3 \leq x_2 \land \mathfrak{N}(x_1, x_2, x_3, x_6)].$$

Take $R_0$ as $A^{(3)}_r(x_1, x_2, r_0)$ and $\Phi^*(x_1, x_2)$ as the formula

$$\Phi(x_1, x_2) \lor [(z) \sim \Phi(x_1, z) \land [(x_2 = \bar{0} \land R_0(x_1, x_2)) \lor (x_2 = \bar{1} \land \sim R_0(x_1, x_2))].$$

**Theorem 2.** $\Phi^*$ strongly represents $\phi$ in $S$.

**Proof.** By Theorem 1 we infer that if $\phi(u) = v$, then $\neg \psi(u, \bar{v})$, so also $\neg \psi^*(u, \bar{v})$. With the help of the same theorem $\neg \psi^*(x, y) \land \Phi^*(x, z) \supset y = z$. It is straightforward to verify that $\neg \psi(Ex)\Phi^*(x, y)$, merely using the logical form of $\Phi^*$, so it remains to be shown that if
\[ \Phi^*(\bar{u}, \bar{v}) \text{ is provable in } S \text{ for some } u, v, \text{ then in fact } \phi(u) = v. \]

Thus suppose that \( \vdash_s \Phi^*(\bar{u}, \bar{v}) \); it suffices to show that \( \vdash_s \Phi(\bar{u}, \bar{v}) \) also.

First note that if \( v > 1 \), then \( \vdash_s \bar{v} \neq 0 \wedge \bar{v} \neq 1 \) by our assumptions about \( S \). But for such \( v \) it easily follows from the form of \( \Phi^*(\bar{u}, \bar{v}) \) that \( \vdash_s \Phi(\bar{u}, \bar{v}) \). Hence we need consider only the cases \( v = 0 \) and \( v = 1 \).

Case \( v = 0 \). In this case we see that \( \vdash_s \sim \Phi(\bar{u}, \bar{0}) \supset \sim R_0(\bar{u}, \bar{0}) \); let \( p \) be the number of a proof. Note that \( M(u, 0, r_0, p) \) holds then, so that \( \vdash_s \exists \lambda (\bar{u}, \bar{0}, r_0, \lambda) \) since \( \exists \lambda \) defines \( M \). Now extend the proof \( p \) by specializing the \( x_4 \) in the definition of \( R_0 \) to \( \bar{p} \), to obtain

\[
(0^*) \quad \vdash_s \sim \Phi(\bar{u}, \bar{0}) \supset (E_{x_6}) [x_6 \leq \bar{p} \wedge \exists \lambda(\bar{u}, \bar{0}, r_0, x_6)].
\]

Now two possibilities arise. First, that \( \vdash_s \sim \Phi(\bar{u}, \bar{0}) \supset \sim R_0(\bar{u}, \bar{0}) \). In this subcase, clearly \( \vdash_s \Phi(\bar{u}, \bar{0}) \) as desired. Otherwise, it is not the case that \( \vdash_s \sim \Phi(\bar{u}, \bar{0}) \supset \sim R_0(\bar{u}, \bar{0}) \), so \( N(u, 0, r_0, q) \) is false for all \( q \), and in particular for \( q \leq p \). Hence for such \( q \), we have \( \vdash_s \exists \lambda(\bar{u}, \bar{0}, r_0, \lambda) \), since \( \exists \lambda \) defines \( N \), and thus \( \vdash_s (x_6) [x_6 \leq \bar{p} \supset \exists \lambda(\bar{u}, \bar{0}, r_0, x_6)] \). From this and \( (0^*) \) we conclude that \( \vdash_s \Phi(\bar{u}, \bar{0}) \).

Case \( v = 1 \). Now we see that \( \vdash_s \sim \Phi(\bar{u}, \bar{1}) \supset \sim R_0(\bar{u}, \bar{1}) \); if \( q \) is the number of a proof, \( N(u, 1, r_0, q) \) holds, so \( \vdash_s \exists \lambda(\bar{u}, \bar{1}, r_0, \lambda) \). Again it is possible that \( \vdash_s \sim \Phi(\bar{u}, \bar{1}) \supset R_0(\bar{u}, \bar{1}) \); and, if so, then \( \vdash_s \Phi(\bar{u}, \bar{1}) \) as desired. Otherwise, there is no proof in \( S \) of \( \sim \Phi(\bar{u}, \bar{1}) \supset R_0(\bar{u}, \bar{1}) \), and so \( M(u, 1, r_0, p) \) fails for all \( p \). In particular, for \( p \leq q \) we have \( \vdash_s \exists \lambda(\bar{u}, \bar{1}, r_0, \lambda) \), and so \( (\text{by (i)}) \) \( \vdash_s \exists \lambda(\bar{u}, \bar{1}, r_0, x_4) \supset (x_4 \leq \bar{q}) \). Now from a proof of \( \exists \lambda(\bar{u}, \bar{1}, r_0, \lambda) \) we can construct one of \( \bar{q} \leq x_4 \supset \bar{q} \leq x_4 \wedge \exists \lambda(\bar{u}, \bar{1}, r_0, x_4) \), so \( \vdash_s \exists \bar{q} \leq x_4 \supset (E_{x_6}) [x_6 \leq x_4 \wedge \exists \lambda(\bar{u}, \bar{1}, r_0, x_6)] \). By \( (\text{ii}) \) we have \( \vdash_s x_4 \leq \bar{q} \forall \bar{q} \leq x_4 \); so by combining this with the above results,

\[
(1^*) \quad \vdash_s \exists \lambda(\bar{u}, \bar{1}, r_0, x_4) \supset (E_{x_6}) [x_6 \leq x_4 \wedge \exists \lambda(\bar{u}, \bar{1}, r_0, x_6)].
\]

Generalize on \( x_4 \) in \( (1^*) \) to obtain \( \vdash_s R_0(\bar{u}, \bar{1}) \), and conclude finally that \( \vdash_s \Phi(\bar{u}, \bar{1}) \) in this case also.

**Bibliography**


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