ON THE VANISHING OF $H^n(X, \mathcal{F})$ FOR AN $n$-DIMENSIONAL VARIETY

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Let $X$ be an irreducible algebraic variety of dimension $n$. Then the cohomology group $H^n(X, \mathcal{F}) = 0$ for all coherent sheaves $F$ if and only if $X$ is nonproper [not complete]. This fact was conjectured by S. Lichtenbaum and proved by A. Grothendieck, in the more general form of the theorem stated below, by means of a delicate argument, which requires an examination both of the residue map and of the relation between local and global duality, [1]. This note gives a more elementary proof of this theorem.

To prove the sufficiency, we reduce to the case $X$ is normal. Here we construct an open affine subset $U$ of $X$ whose complement $Y$ is again irreducible and nonproper, and we consider the canonical exact sequence

$$0 \to \mathcal{F} \to i_* i^* \mathcal{F} \to \text{Coker } \psi \to 0$$

where $i: U \to X$ is the inclusion map. As $H^q(X, i_* i^* \mathcal{F}) = 0$ for $q > 0$, to be able to finish by induction on $n = \dim X$, in Remark 1 we strengthen the theorem to the form in which $X$ is a closed subscheme of $Z$ and $F$ is a quasi-coherent $\mathcal{O}_Z$-module. We start the induction with $n = 1$, here $X$ is an affine curve. However if we start the induction with $n = 0$, the proof yields $H^q(X, \mathcal{F}) = 0$ for $q > n$, $X$ proper or not.

To prove the necessity, we first reduce to the case $X$ is projective by taking a Chow cover of $X$ and applying the Leray spectral sequence. Then we prove $H^n(X, \mathcal{O}_X(-m)) \neq 0$ for all $m \gg 0$ by induction on $n$.

Theorem. Let $X$ be an $n$-dimensional algebraic scheme over the field $k$. Then for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, $H^n(X, \mathcal{F})$ is a finite dimensional vector space over $k$. Furthermore, the following conditions are equivalent:

(i) All irreducible components of $X$ of dimension $n$ are nonproper.

(ii) $H^n(X, \mathcal{F}) = 0$ for all coherent $\mathcal{O}_X$-modules $\mathcal{F}$. Moreover, if $X$ is quasi-projective and $\mathcal{O}_X(1)$ is a very ample $\mathcal{O}_X$-module, then (i) and (ii) are also equivalent to

(iii) $H^n(X, \mathcal{O}_X(-m)) = 0$ for all $m \gg 0$.

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Remark 1. Suppose \( X \) is a closed subprescheme of a noetherian prescheme \( Z \). Then for any integer \( n \), the following conditions are equivalent:

(a) \( H^n(X, \mathcal{F}) = 0 \) for all coherent \( O_X \)-modules \( \mathcal{F} \).

(b) \( H^n(Z, \mathcal{F}) = 0 \) for all quasi-coherent \( O_Z \)-modules \( \mathcal{F} \) with support in \( X \).

Indeed the implication (b) \( \Rightarrow \) (a) is clear. Conversely let \( \mathcal{F} \) be a quasi-coherent \( O_Z \)-module with support in \( X \). \( \mathcal{F} \) is the direct limit of its coherent submodules \( \mathcal{G} \) (cf. [2(a)], and \( H^n(Z, \mathcal{F}) \) is the direct limit of the \( H^n(Z, \mathcal{G}) \), by [3]; hence, we may assume \( \mathcal{F} \) is coherent.

Let \( X' \) be the subprescheme of \( Z \) defined by the annihilator of \( \mathcal{F} \). Then \( \mathcal{F} \) is a coherent \( O_X \)-module. Further the reduction \( X'' \) of \( X' \) is a subprescheme of \( X \) because its underlying space, which is the support of \( \mathcal{F} \), is contained in \( X \). Therefore (a) implies that the set \( K' \) of coherent \( O_X \)-modules \( \mathcal{F} \) such that \( H^n(X', \mathcal{F}) = 0 \) contains every coherent \( O_X \)-module; hence, it contains every coherent \( O_{X''} \)-module by the following lemma.

Lemma 1. Let \( X \) be a noetherian prescheme, and let \( K' \) be a set of coherent \( O_X \)-modules which satisfies the following two conditions:

1. For every exact sequence \( 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0 \) of coherent \( O_X \)-modules such that \( \mathcal{F}', \mathcal{F}'' \in K' \), also \( \mathcal{F} \in K' \).

2. For every irreducible component \( Y \) of \( X \) given its unique induced reduced structure and for every coherent \( O_Y \)-module \( \mathcal{F} \), \( \mathcal{F} \in K' \).

Then \( K' \) is the set of "all" coherent \( O_X \)-modules.

Indeed let \( Y_1, \ldots, Y_m \) be the irreducible components of \( X \), and let \( \mathcal{g}_1, \ldots, \mathcal{g}_m \) be their defining sheaves of ideals. Then \( (\mathcal{g}_1 \cdots \mathcal{g}_m)^k = 0 \) for some integer \( k \). Now given a coherent \( O_X \)-module \( \mathcal{F} \), set \( \mathcal{F}_{ij} = \mathcal{g}_1^{i+1} \cdots \mathcal{g}_j^{j+1} \cdots \mathcal{g}_m^m \mathcal{F} \) for \( i = 0, \ldots, k \) and \( j = 1, \ldots, m \). The \( \mathcal{F}_{ij} \) ordered lexicographically, filter \( \mathcal{F} \). Their successive quotients are \( O_{Y_l} \)-modules for suitable \( l \), and so are in \( K' \) by (2). By (1) and by induction \( \mathcal{F} \in K' \).

Remark 2. Let \( Y \) be a locally noetherian prescheme, \( f: X \rightarrow Y \) a separated morphism of finite type, \( y \) a point of \( Y \), and \( n \) the dimension of \( f^{-1}(y) \). Then it is not true in general that

\[
(R^q f_* \mathcal{F})_y = 0
\]

for all coherent \( O_X \)-modules \( \mathcal{F} \) and all \( q > n \), so we cannot expect a relative form of the theorem.

For example, let \( Y \) be a nonsingular variety of dimension \( r > 2 \), \( y \) a closed point of \( Y \), \( X = Y - \{y\} \), and \( f: X \rightarrow Y \) the inclusion. Then via
local cohomology we easily compute that \( R^{r-1} f_* \mathcal{O}_X \) is the injective hull of \( k(y) \) supported at \( y \).

On the other hand, (*) does hold if \( f \) is proper, [2(b)].

Returning to the theorem, to prove \( H^s(X, \mathfrak{F}) \) is finite dimensional, we may assume \( X \) is reduced and irreducible by Lemma 1. Then if \( X \) is proper, \( H^s(X, \mathfrak{F}) \) is finite dimensional by the finiteness theorem [2(c)]; if \( X \) is nonproper, \( H^s(X, \mathfrak{F}) = 0 \) by the implication (i)\(\Rightarrow\)(ii) proved next.

To prove (i)\(\Rightarrow\)(ii), again by Lemma 1, we may assume \( X \) is reduced and irreducible. We may also assume \( \mathfrak{F} \) is torsion free. For let \( \mathfrak{I} \) be the torsion submodule of \( \mathfrak{F} \), and set \( \mathfrak{G} = \mathfrak{F}/\mathfrak{I} \). Then \( \mathfrak{G} \) is torsion free, and \( H^s(X, \mathfrak{G}) \cong H^s(X, \mathfrak{F}) \) because \( H^s(X, \mathfrak{I}) = 0 \). Finally we may assume \( X \) is normal by the following beautiful argument due to Grothendieck [2(d)].

Let \( X' \) be the normalization of \( X \) in its function field, and let \( f = (\Psi, \theta) : X' \to X \) be the canonical morphism. \( \theta : \mathcal{O}_X \to f_* \mathcal{O}_{X'} \) is an isomorphism on some open set \( U \) because \( f \) is birational. So \( \theta \) induces a map

\[
v : \mathfrak{G} = \text{Hom}_{\mathcal{O}_X} (f_* \mathcal{O}_{X'}, \mathfrak{F}) \to \text{Hom}_{\mathcal{O}_X} (\mathcal{O}_X, \mathfrak{F}) = \mathfrak{F},
\]

which is also an isomorphism on \( U \). The kernel of \( v \), \( \text{Hom}_{\mathcal{O}_X} (\text{Coker } \theta, \mathfrak{G}) \) is zero because \( \mathfrak{F} \) is torsion free and because \( \text{Coker } \theta \) is torsion, \( \theta \) being an isomorphism on \( U \). Thus we have the exact sequence

\[
0 \to \mathfrak{G} \to \mathfrak{F} \to \text{Coker } v \to 0,
\]

and it suffices to show \( H^s(X, \mathfrak{G}) = H^s(X, \text{Coker } v) = 0 \).

\( \text{Coker } v \) is torsion because \( v \) is an isomorphism on \( U \); hence \( H^s(X, \text{Coker } v) = 0 \). On the other hand, \( \mathfrak{G} \) is a coherent \( f_* \mathcal{O}_{X'} \)-module. Since \( X' \) is affine over \( X \), \( \mathfrak{G} = f_* \mathfrak{G}' \) for some coherent \( \mathcal{O}_{X'} \)-module \( \mathfrak{G}' \), and \( H^s(X, \mathfrak{G}) = H^s(X', \mathfrak{G}') \). But \( X' \) is nonproper because \( X \) is, and \( X' \) is normal.

**Lemma 2.** Let \( X \) be an irreducible normal algebraic scheme of dimension at least 2. Then there exists a closed irreducible subscheme \( Y \) of \( X \) such that \( X - Y \) is affine. Further \( Y \) is nonproper if \( X \) is.

Indeed let \((X', f)\) be a Chow cover of \( X \): \( X' \) is reduced and quasi-projective, \( f : X' \to X \) is proper and birational. Let \( E' \) be the exceptional locus of \( f \), the closed set of points \( x \in X' \) such that \( \dim f^{-1}(f(x)) \geq 1 \), or such that \( f \) is not birational at \( x \), equivalently \( X \) being normal. Let \( X'' \) be the closure of \( X' \) in some projective space, \( E'' \) the closure of \( E \) in \( X'' \), and set \( Z = E'' \cup (X'' - X') \). Blowing up \( Z \), we may assume \( Z \) is a Cartier divisor.
Let $Y''$ be an irreducible hypersurface section of $X''$, and let $Y = f(Y'' \cap X')$. $Y$ is closed irreducible, and $X - Y$ is isomorphic to $X'' - (Z \cup Y'')$ because $Y''$ meets the fibre $f^{-1}(x)$ through every \( x \in E' \) and because $f$ is an isomorphism off $E'$. But the Cartier divisor $Z + mY''$, $m > 0$, is very ample; hence, $X'' - (Z \cup Y'')$ is affine.

If $X$ is nonproper, then $X'$ is also nonproper. So $X'' - X'$ is nonempty and of pure codimension 1. Hence $Y'' \cap X'$ and $Y = f(Y'' \cap X')$ are nonproper.

To finish proving (i) $\Rightarrow$ (ii), we proceed by induction on $n$, the dimension of $X$. $n \neq 0$, for otherwise $X$ would be proper. If $n = 1$, $X$ is affine; hence certainly $H^n(X, F) = 0$. Assume then $n \geq 2$.

Apply Lemma 2; let $U = X - Y$ and $i: U \to X$. Consider the canonical map $v: \mathcal{F} \to i_*i^*\mathcal{F}$. $v$ is an isomorphism on $U$; hence Ker $v$ and Coker $v$ are torsion with support in $Y$. Because $\mathcal{F}$ is torsion free, Ker $v$ is zero, and we have the exact sequence

$$0 \to \mathcal{F} \to i_*i^*\mathcal{F} \to \text{Coker } v \to 0.$$  

But $H^{n-1}(X, \text{Coker } v) = 0$ by induction and by Remark 1, while $H^n(X, i_*i^*\mathcal{F}) = 0$ by the following lemma.

**Lemma 3.** Let $X$ be a scheme, $U$ an affine subscheme, $i: U \to X$ the inclusion. Then for any quasi-coherent sheaf $\mathcal{F}$ on $U$, $H^q(X, i_*i^*\mathcal{F}) = 0$ for all $q > 0$.

Indeed since $X$ is separated, $i$ is an affine morphism. Hence $H^q(X, i_*\mathcal{F}) = 0$ for all $q > 0$.

The implication (ii) $\Rightarrow$ (iii) of the theorem is trivial. Conversely for all coherent $\mathcal{F}$ and all $m > 0$, there exists a surjection $0 \to \mathcal{F} \to 0$; hence (iii) $\Rightarrow$ (ii).

To prove (ii) $\Rightarrow$ (i), we assume $X$ is irreducible and proper, and we construct a coherent $\mathcal{O}_X$-module $\mathcal{F}$ such that $H^n(X, \mathcal{F}) \neq 0$. First we reduce to the case $X$ is quasi-projective by applying the following lemma to a Chow cover $(X', f)$ of $X$.

**Lemma 4.** Let $X$ be a noetherian prescheme of dimension $n$, $f: X' \to X$ a proper birational map, $\mathcal{F}$ a coherent $\mathcal{O}_{X'}$-module. Then $H^n(X', \mathcal{F}) \neq 0$ implies $H^n(X, f_*\mathcal{F}) \neq 0$.

Indeed for $q = 0, 1, \ldots, n - 1$ let $Z_q$ be the closed set of points $x \in X$ such that dim $f^{-1}(x) \geq n - q$. By Remark 2, $R^{n-q}f_*\mathcal{F}$ has support in $Z_q$. But dim $f^{-1}(Z_q) \leq n - 1$ because $f$ is an isomorphism on an open set. Hence dim $Z_q \leq (n - 1) - (n - q) = q - 1$. Therefore $H^q(X, R^{n-q}f_*\mathcal{F}) = H^q(Z_q, R^{n-q}f_*\mathcal{F}) = 0$, and the Leray spectral sequence yields a surjection.
\[
H^n(X, f_*\mathcal{F}) \to H^n(X', \mathcal{F}) \to 0,
\]

completing the proof of the lemma.

Finally, when \(X\) is projective, we prove \(H^n(X, \mathcal{O}_X(-m)) \neq 0\) for \(m \gg 0\). We simply bound \(\dim H^q(X, \mathcal{O}_X(-m))\) by a polynomial \(P_X(m)\) of degree \(\leq n - 1\), for \(q \leq n - 1\) and \(m \geq 0\). For then \(\dim H^n(X, \mathcal{O}_X(-m)) \geq \chi(\mathcal{O}_X(m)) - n P_X(m) = (\deg X/n!)m^n + \cdots\). We construct \(P_X(m)\) by induction on \(n\). When \(n = 0\), we take \(P(m) = 0\). When \(n > 0\), we find a hyperplane section \(H\) of \(X\) which avoids \(\text{Ass} X\). Then the sequence

\[
0 \to \mathcal{O}_X(-m - 1) \to \mathcal{O}_X(-m) \to \mathcal{O}_H(-m) \to 0
\]

is exact and yields \(\dim H^q(X, \mathcal{O}_X(-m - 1)) - \dim H^q(X, \mathcal{O}_X(-m)) \leq \dim H^q(H, \mathcal{O}_H(-m)) \leq P_H(m)\); whence we may construct \(P_X\) from \(P_H\) and \(\dim H^q(X, \mathcal{O}_X)\).

References

   (a) (I, 9.4.9, Corollaire).
   (b) (III, 4.2.2, Corollaire).
   (c) III, 3.2.3, Corollaire.
   (d) II, 6.7, Chevalley's theorem.