ON THE VANISHING OF $H^n(X, \mathfrak{F})$ FOR AN n-DIMENSIONAL VARIETY

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Let X be an irreducible algebraic variety of dimension n. Then the cohomology group $H^n(X, \mathfrak{F})=0$ for all coherent sheaves F if and only if X is nonproper [=not complete]. This fact was conjectured by S. Lichtenbaum and proved by A. Grothendieck, in the more general form of the theorem stated below, by means of a delicate argument, which requires an examination both of the residue map and of the relation between local and global duality, [**1**]. This note gives a more elementary proof of this theorem.

To prove the sufficiency, we reduce to the case X is normal. Here we construct an open affine subset U of X whose complement Y is again irreducible and nonproper, and we consider the canonical exact sequence

$$0 \longrightarrow \mathfrak{F} \xrightarrow{v} i_*i^*\mathfrak{F} \longrightarrow \text{Coker } v \longrightarrow 0$$

where $i: U \to X$ is the inclusion map. As $H^q(X, i_*i^*\mathfrak{F})=0$ for $q>0$, to be able to finish by induction on $n=\dim X$, in Remark 1 we strengthen the theorem to the form in which X is a closed subprescheme of Z and F is a quasi-coherent $\mathcal{O}_Z$-module. We start the induction with $n=1$, here X is an affine curve. However if we start the induction with $n=0$, the proof yields $H^q(X, \mathfrak{F})=0$ for $q>n$, X proper or not.

To prove the necessity, we first reduce to the case X is projective by taking a Chow cover of X and applying the Leray spectral sequence. Then we prove $H^n(X, \mathcal{O}_X(-m))\neq 0$ for all $m \gg 0$ by induction on n.

THEOREM. *Let X be an n-dimensional algebraic scheme over the field k. Then for any coherent $\mathcal{O}_X$-module $\mathfrak{F}$, $H^n(X, \mathfrak{F})$ is a finite dimensional vector space over k. Furthermore, the following conditions are equivalent;*

(i) *All irreducible components of X of dimension n are nonproper.*

(ii) $H^n(X, \mathfrak{F})=0$ *for all coherent $\mathcal{O}_X$-modules $\mathfrak{F}$. Moreover, if X is quasi-projective and $\mathcal{O}_X(1)$ is a very ample $\mathcal{O}_X$-module, then* (i) *and* (ii) *are also equivalent to*

(iii) $H^n(X, \mathcal{O}_X(-m))=0$ *for all* $m \gg 0$.

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REMARK 1. Suppose X is a closed subprescheme of a noetherian prescheme Z. Then for any integer n, the following conditions are equivalent:

(a) $H^n(X, \mathfrak{F})=0$ *for all coherent* $\mathcal{O}_X$*-modules* $\mathfrak{F}$.

(b) $H^n(Z, \mathfrak{F})=0$ *for all quasi-coherent* $\mathcal{O}_Z$*-modules* $\mathfrak{F}$ *with support in* X.

Indeed the implication (b)$\Rightarrow$(a) is clear. Conversely let $\mathfrak{F}$ be a quasi-coherent $\mathcal{O}_Z$-module with support in X. $\mathfrak{F}$ is the direct limit of its coherent submodules $\mathcal{G}$ (cf. [2(a)], and $H^n(Z, \mathfrak{F})$ is the direct limit of the $H^n(Z, \mathcal{G})$, by [3]; hence, we may assume $\mathfrak{F}$ is coherent.

Let X' be the subprescheme of Z defined by the annihilator of $\mathfrak{F}$. Then $\mathfrak{F}$ is a coherent $\mathcal{O}_X$-module. Further the reduction X'' of X' is a subprescheme of X because its underlying space, which is the support of $\mathfrak{F}$, is contained in X. Therefore (a) implies that the set K' of coherent $\mathcal{O}_{X'}$-modules $\mathfrak{F}$ such that $H^n(X', \mathfrak{F})=0$ contains every coherent $\mathcal{O}_{X''}$-module; hence, it contains every coherent $\mathcal{O}_{X'}$-module by the following lemma.

LEMMA 1. *Let X be a noetherian prescheme, and let K' be a set of coherent $\mathcal{O}_X$-modules which satisfies the following two conditions:*

(1) *For every exact sequence* $0\to\mathfrak{F}'\to\mathfrak{F}\to\mathfrak{F}''\to 0$ *of coherent* $\mathcal{O}_X$*-modules such that* $\mathfrak{F}', \mathfrak{F}''\in K'$, *also* $\mathfrak{F}\in K'$.

(2) *For every irreducible component Y of X given its unique induced reduced structure and for every coherent $\mathcal{O}_Y$-module $\mathfrak{F}$, $\mathfrak{F}\in K'$.*

Then K' is the set of "all" coherent $\mathcal{O}_X$-modules.

Indeed let $Y_1, \cdots, Y_m$ be the irreducible components of X, and let $\mathcal{g}_1, \cdots, , \mathcal{g}_m$ be their defining sheaves of ideals. Then $(\mathcal{g}_1 \cdots \mathcal{g}_m)^k=0$ for some integer k. Now given a coherent $\mathcal{O}_X$-module $\mathfrak{F}$, set $\mathfrak{F}_{ij}=\mathcal{g}_1^{i+1}\cdots \mathcal{g}_{j-1}^{i+1}\cdots \mathcal{g}_m^{i}\cdot\mathfrak{F}$ for $i=0, \cdots, k$ and $j=1, \cdots, m$. The $\mathfrak{F}_{ij}$ ordered lexicographically, filter $\mathfrak{F}$. Their successive quotients are $\mathcal{O}_{Y_l}$-modules for suitable l, and so are in K' by (2). By (1) and by induction $\mathfrak{F}\in K'$.

REMARK 2. Let Y be a locally noetherian prescheme, $f\colon X\to Y$ a separated morphism of finite type, y a point of Y, and n the dimension of $f^{-1}(y)$. Then it is not true in general that

$$(R^qf_*\mathfrak{F})_y = 0 \tag{*}$$

for all coherent $\mathcal{O}_X$-modules $\mathfrak{F}$ and all $q>n$, so we cannot expect a relative form of the theorem.

For example, let Y be a nonsingular variety of dimension $r>2$, y a closed point of Y, $X=Y-\{y\}$, and $f\colon X\to Y$ the inclusion. Then via local cohomology we easily compute that $R^{r-1}f_*\mathcal{O}_X$ is the injective hull of $k(y)$ supported at y.

On the other hand, (*) does hold if f is *proper*, [**2**(b)].

Returning to the theorem, to prove $H^n(X, \mathfrak{F})$ is finite dimensional, we may assume X is reduced and irreducible by Lemma 1. Then if X is proper, $H^n(X, \mathfrak{F})$ is finite dimensional by the finiteness theorem [**2**(c)]; if X is nonproper, $H^n(X, \mathfrak{F})=0$ by the implication (i)$\Rightarrow$(ii) proved next.

To prove (i)$\Rightarrow$(ii), again by Lemma 1, we may assume X is reduced and irreducible. We may also assume $\mathfrak{F}$ is torsion free. For let $\mathfrak{J}$ be the torsion submodule of $\mathfrak{F}$, and set $\mathcal{G}=\mathfrak{F}/\mathfrak{J}$. Then $\mathcal{G}$ is torsion free, and $H^n(X, \mathfrak{F})\xrightarrow{\sim}H^n(X, \mathcal{G})$ because $H^n(X, \mathfrak{J})=0$. Finally we may assume X is normal by the following beautiful argument due to Grothendieck [**2**(d)].

Let X' be the normalization of X in its function field, and let $f=(\Psi, \theta): X'\to X$ be the canonical morphism. $\theta: \mathcal{O}_X\to f_*\mathcal{O}_{X'}$ is an isomorphism on some open set U because f is birational. So θ induces a map

$$v: \mathcal{G} = \operatorname{Hom}_{\mathcal{O}_X}(f_*\mathcal{O}_{X'}, \mathfrak{F}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathfrak{F}) = \mathfrak{F},$$

which is also an isomorphism on U. The kernel of v, $\operatorname{Hom}_{\mathcal{O}_X}(\operatorname{Coker}\theta, \mathfrak{F})$ is zero because $\mathfrak{F}$ is torsion free and because $\operatorname{Coker}\theta$ is torsion, θ being an isomorphism on U. Thus we have the exact sequence

$$0\to \mathcal{G}\to\mathfrak{F}\to \operatorname{Coker} v\to 0,$$

and it suffices to show $H^n(X, \mathcal{G})=H^n(X, \operatorname{Coker} v)=0$.

$\operatorname{Coker} v$ is torsion because v is an isomorphism on U; hence $H^n(X, \operatorname{Coker} v)=0$. On the other hand, $\mathcal{G}$ is a coherent $f_*\mathcal{O}_{X'}$-module. Since X' is affine over X, $\mathcal{G}=f_*\mathcal{G}'$ for some coherent $\mathcal{O}_{X'}$-module $\mathcal{G}'$, and $H^n(X, \mathcal{G})=H^n(X', \mathcal{G}')$. But X' is nonproper because X is, and X' is normal.

LEMMA 2. *Let X be an irreducible normal algebraic scheme of dimension at least 2. Then there exists a closed irreducible subscheme Y of X such that $X-Y$ is affine. Further Y is nonproper if X is.*

Indeed let (X', f) be a Chow cover of X: X' is reduced and quasi-projective, $f: X'\to X$ is proper and birational. Let E' be the exceptional locus of f, the closed set of points $x\in X'$ such that $\dim_x f^{-1}f(x)\geqq 1$, or such that f is not biregular at x, equivalently X being normal. Let X'' be the closure of X' in some projective space, E'' the closure of E in X'', and set $Z=E''\cup(X''-X')$. Blowing up Z, we may assume Z is a Cartier divisor.

Let Y'' be an irreducible hypersurface section of X'', and let $Y=f(Y''\cap X')$. Y is closed irreducible, and $X-Y$ is isomorphic to $X''-(Z\cup Y'')$ because Y'' meets the fibre $f^{-1}f(x)$ through every $x\in E'$ and because f is an isomorphism off E'. But the Cartier divisor $Z+mY''$, $m\gg 0$, is very ample; hence, $X''-(Z\cup Y'')$ is affine.

If X is nonproper, then X' is also nonproper. So $X''-X'$ is nonempty and of pure codimension 1. Hence $Y''\cap(X''-X')$ is also nonempty. Therefore $Y''\cap X'$ and $Y=f(Y''\cap X')$ are nonproper.

To finish proving (i)$\Rightarrow$(ii), we proceed by induction on n, the dimension of X. $n\neq 0$, for otherwise X would be proper. If $n=1$, X is affine; hence certainly $H^n(X, F)=0$. Assume then $n\geqq 2$.

Apply Lemma 2; let $U=X-Y$ and i: $U\rightarrow X$. Consider the canonical map v: $\mathfrak{F}\rightarrow i_*i^*\mathfrak{F}$. v is an isomorphism on U; hence Ker v and Coker v are torsion with support in Y. Because $\mathfrak{F}$ is torsion free, Ker v is zero, and we have the exact sequence

$$0\rightarrow \mathfrak{F}\rightarrow i_*i^*\mathfrak{F}\rightarrow \text{Coker } v\rightarrow 0.$$

But $H^{n-1}(X, \text{Coker } v)=0$ by induction and by Remark 1, while $H^n(X, i_*i^*\mathfrak{F})=0$ by the following lemma.

LEMMA 3. *Let X be a scheme, U an affine subscheme, i: $U\rightarrow X$ the inclusion. Then for any quasi-coherent sheaf $\mathfrak{F}$ on U, $H^q(X, i_*\mathfrak{F})=0$ for all $q>0$.*

Indeed since X is separated, i is an affine morphism. Hence $H^q(X, i_*\mathfrak{F})=0$ for all $q>0$.

The implication (ii)$\Rightarrow$(iii) of the theorem is trivial. Conversely for all coherent $\mathfrak{F}$ and all $m\gg 0$, there exists a surjection $\mathcal{O}_X(-m)\rightarrow\mathfrak{F}\rightarrow 0$; hence (iii)$\Rightarrow$(ii).

To prove (ii)$\Rightarrow$(i), we assume X is irreducible and proper, and we construct a coherent $\mathcal{O}_X$-module $\mathfrak{F}$ such that $H^n(X, \mathfrak{F})\neq 0$. First we reduce to the case X is quasi-projective by applying the following lemma to a Chow cover (X', f) of X.

LEMMA 4. *Let X be a noetherian prescheme of dimension n, f: $X'\rightarrow X$ a proper birational map, $\mathfrak{F}$ a coherent $\mathcal{O}_{X'}$-module. Then $H^n(X', \mathfrak{F})\neq 0$ implies $H^n(X, f_*\mathfrak{F})\neq 0$.*

Indeed for $q=0, 1, \cdots, n-1$ let Z_q be the closed set of points $x\in X$ such that $\dim f^{-1}(x)\geqq n-q$. By Remark 2, $R^{n-q}f_*\mathfrak{F}$ has support in Z_q. But $\dim f^{-1}(Z_q)\leqq n-1$ because f is an isomorphism on an open set. Hence $\dim Z_q\leqq(n-1)-(n-q)=q-1$. Therefore $H^q(X, R^{n-q}f_*\mathfrak{F})$ $=H^q(Z_q, R^{n-q}f_*\mathfrak{F})=0$, and the Leray spectral sequence yields a surjection

$$H^n(X, f_*\mathfrak{F}) \to H^n(X', \mathfrak{F}) \to 0,$$

completing the proof of the lemma.

Finally, when X is projective, we prove $H^n(X, \mathcal{O}_X(-m)) \neq 0$ for $m \gg 0$. We simply bound dim $H^q(X, \mathcal{O}_X(-m))$ by a polynomial $P_X(m)$ of degree $\leqq n-1$, for $q \leqq n-1$ and $m \geqq 0$. For then dim $H^n(X, \mathcal{O}_X(-m)) \geqq \chi(\mathcal{O}_X(m)) - n\, P_X(m) = (\deg X/n!)m^n + \cdots$. We construct $P_X(m)$ by induction on n. When $n=0$, we take $P(m)=0$. When $n > 0$,we find a hyperplane section H of X which avoids Ass X. Then the sequence

$$0 \to \mathcal{O}_X(-m-1) \to \mathcal{O}_X(-m) \to \mathcal{O}_H(-m) \to 0$$

is exact and yields dim $H^q(X, \mathcal{O}_X(-m-1))$ − dim $H^q(X, \mathcal{O}_X(-m)) \leqq$ dim $H^q(H, \mathcal{O}_H(-m)) \leqq P_H(m)$; whence we may construct P_X from P_H and dim $H^q(X, \mathcal{O}_X)$.

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(a) (I, 9.4.9, Corollaire).

(b) (III, 4.2.2, Corollaire).

(c) III, 3.2.3, Corollaire.

(d) II, 6.7, Chevalley's theorem.

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