NONUNIQUENESS OF EXTREMAL KERNELS

JAMES G. CAUGHRAN

Given a "kernel" $k = k(e^{i\theta}) \in L^q$, then

$$\Phi(f) = \frac{1}{2\pi i} \int_{|z|=1} f(z)k(z)dz$$

is a bounded linear functional on $H^p$ ($1/p + 1/q = 1$) with norm $||\Phi|| \leq ||k||_q$. Two kernels induce the same linear functional, $k_1 \sim k_2$, if and only if they differ by an $H^q$ function. It can be shown that the duality relation, $||\Phi|| = \inf \{||k_1||_q : k_1 \sim k_2\}$ holds for $1 \leq p \leq \infty$. One may ask whether an extremal kernel exists, i.e., a kernel $k_1 \sim k$ such that $||\Phi|| = ||k_1||_q$, and if so, whether it is unique.

By functional analytic methods, Havinson [1] and Rogosinski and Shapiro [2] showed that an extremal kernel always exists for $1 \leq p \leq \infty$ and is unique for $1 < p \leq \infty$. However, Rogosinski and Shapiro constructed a counterexample to show that it need not be unique for $p = 1$. As their example to show nonuniqueness for $p = 1$ was rather complicated, we present a simplification of their example.

Let

$$k(e^{i\theta}) = \begin{cases} 1, & 0 \leq \theta < \pi/2, \\ -1, & \pi/2 \leq \theta < \pi, \\ 0, & \pi \leq \theta < 2\pi. \end{cases}$$

Rogosinski and Shapiro showed that this kernel induces a functional $\Phi$ on $H^1$ with $||\Phi|| = 1$, and thus $k$ is an extremal kernel. To show this extremal kernel is not unique, it suffices to produce an $H^1$ function $h \neq 0$ for which $||k + h||_\infty = 1$.

Let $h$ be the conformal mapping sending the unit disk to the upper half-disk $|w| < 1$, $\text{Im } w > 0$, with $h(1) = -1$, $h(i) = 0$, and $h(-1) = 1$. Then $h \in H^\infty$, and

$$-1 \leq h(e^{i\theta}) < 0, \quad 0 \leq \theta < \pi/2$$

$$0 \leq h(e^{i\theta}) < 1, \quad \pi/2 \leq \theta < \pi$$

$$|h(e^{i\theta})| = 1, \quad \pi \leq \theta < 2\pi.$$

Thus $|k(e^{i\theta}) + h(e^{i\theta})| \leq 1$, and $h$ is the desired function.

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The Koebe function $z/(1+z)^2$ is positive everywhere on $|z|=1$, $z \neq -1$, and lies in the Hardy class $H^p$ for every $p < 1/2$. We show that this behavior is extreme by proving the following.

**Theorem.** If $f(z) \in H^{1/2}$ and $f(z) \geq 0$ a.e. on $|z|=1$ then $f(z)$ is a constant.

**Proof.** We may assume that $f(z)$ is not identically 0. If $B(z)$ denotes the Blaschke product for the zeros of $f(z)$ then, as usual, we can write

$$f(z) = B(z)F^2(z), \quad F(z) \in H^1. \quad (1)$$

We write the condition $f(z) \geq 0$ as the equation $f(z) = |f(z)|$ and conclude from (1) that

$$B(z)F^2(z) = |F^2(z)| \quad \text{a.e. on } |z| = 1. \quad (2)$$

Since $f(z)$ is not identically 0 it follows that $F(z)$ is nonzero a.e. on $|z| = 1$. Thus we may divide (2) by $F(z)$ and obtain

$$B(z)F(z) = \overline{F(z)} \quad \text{a.e. on } |z| = 1. \quad (3)$$

But the left side of (3) is $H^1$ and so has all negative Fourier coefficients 0, the right side is conjugate $H^1$ and so has all positive Fourier coefficients 0!.

Thus only the constant term remains and we conclude that both sides are constants. This is to say $B(z)F(z)$ and $F(z)$ are both constants and so indeed $f(z) = (B(z)F(z))$. $F(z)$ is a constant.

**University of Connecticut and**

**Yeshiva University**

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