ON TYPE I $C^*$-ALGEBRAS

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1. Introduction. Recently, the author [4] proved the equivalence of type I $C^*$-algebras and GCR $C^*$-algebras without the assumption of separability. On the other hand, for separable type I $C^*$-algebras, we have a simpler criterion as follows: a separable $C^*$-algebra $\mathfrak{A}$ is of type I if and only if every irreducible image contains a nonzero compact operator.

It has been open whether or not this remains true when $\mathfrak{A}$ is not separable (cf. [1], [2], [3]).

In the present paper, we shall show that a $C^*$-algebra $\mathfrak{A}$ is GCR if and only if every irreducible image contains a nonzero compact operator, so that by the author's previous theorem [4], the above problem is affirmative for arbitrary $C^*$-algebra.

2. Theorem. In this section, we shall show the following theorem.

Theorem. A $C^*$-algebra $\mathfrak{A}$ is of type I if and only if every irreducible image contains a nonzero compact operator.

Proof. Suppose that a $C^*$-algebra $\mathfrak{A}$ is of type I, then it is GCR and so every irreducible image contains a nonzero compact operator (cf. [1], [2], [3], [4]).

Conversely suppose that every irreducible image of $\mathfrak{A}$ contains a nonzero compact operator. It is enough to assume that $\mathfrak{A}$ has the unit $I$. We shall assume that $\mathfrak{A}$ is not of type I. Then it is not GCR; then there is a separable non-type I $C^*$-subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ (cf. [2], [4]). Take a pure state $\phi$ on $\mathfrak{B}$ such that the image of $\mathfrak{B}$ under the irreducible $*$-representation $\{U_\phi, \mathfrak{H}_\phi\}$ of $\mathfrak{B}$ constructed via $\phi$ does not contain any nonzero compact operator, where $\mathfrak{H}_\phi$ is a Hilbert space.

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Let $\mathcal{E}$ be the set of all pure states $\psi$ on $\mathfrak{A}$ such that $\psi = \phi$ on $\mathfrak{B}$. We shall define a partial ordering $\prec$ in $\mathcal{E}$ in the following. Take $\psi \in \mathcal{E}$, and 
\{\pi_\psi, \mathfrak{H}_\psi\} be the irreducible *-representation of $\mathfrak{A}$ constructed via $\psi$, then $\pi_\psi(\mathfrak{H})$ contains a nonzero compact operator; hence $\pi_\psi(\mathfrak{H})$ contains the algebra $C(\mathfrak{H}_\psi)$ of all compact operators (cf. [1]). Let $\mathfrak{D}(\psi) = \pi_\psi^{-1}(C(\mathfrak{H}_\psi))$, then $\mathfrak{D}(\psi)$ is an ideal of $\mathfrak{A}$. For $\psi_1, \psi_2 \in \mathcal{E}$, we shall define the order as follows: $\psi_1 \prec \psi_2$ if $\mathfrak{D}(\psi_1) \subset \mathfrak{D}(\psi_2)$. Let $\{\psi_{\alpha}| \alpha \in \Pi\}$ be a linearly ordered subset of $\mathcal{E}$, and let $\mathfrak{D}$ be the uniform closure of $U_{\alpha \in \Pi} \mathfrak{D}(\psi_{\alpha})$, then $\mathfrak{D}$ is an ideal of $\mathfrak{A}$. Let $\mathfrak{F}$ be the kernel of the representation $\{U_{\phi}, \mathfrak{H}_\phi\}$ of $\mathfrak{B}$. First of all we shall show that $\mathfrak{B} \cap \mathfrak{D} \subset \mathfrak{F}$. Suppose that $\mathfrak{B} \cap \mathfrak{D} \subset \mathfrak{F}$, then there is an element $b \in (\mathfrak{B} \cap \mathfrak{D}) \cap \mathfrak{F}^c$ and $b_n \in \mathfrak{D}(\psi_{\alpha})$ for $n = 1, 2, 3, \ldots$ such that $\|U_\phi(b)\| = 1$ and $\|b - b_n\| < 1/n$ for $n = 1, 2, 3, \ldots$, where $\mathfrak{F}^c$ is the complement of $\mathfrak{F}$ in $\mathfrak{B}$.

Take the representation $\{\pi_{\psi_{\alpha}}, \mathfrak{H}_{\psi_{\alpha}}\}$ of $\mathfrak{A}$, then $\|\pi_{\psi_{\alpha}}(b) - \pi_{\psi_{\alpha}}(b_n)\| < 1/n$.

Let $[\pi_{\psi_{\alpha}}(\mathfrak{B})]I_{\psi_{\alpha}}$ be the closed subspace generated by $\pi_{\psi_{\alpha}}(\mathfrak{B})I_{\psi_{\alpha}}$, where $I_{\psi_{\alpha}}$ is the image of $I$ in $\mathfrak{H}_{\psi_{\alpha}}$, and let $E_n'$ be the orthogonal projection of $\mathfrak{H}_{\psi_{\alpha}}$ onto $[\pi_{\psi_{\alpha}}(\mathfrak{B})]I_{\psi_{\alpha}}$. Then the representation $y \mapsto \pi_{\psi_{\alpha}}(y)E_n'$ for $y \in \mathfrak{B}$ is equivalent to $\{U_\phi, \mathfrak{H}_\phi\}$.

On the other hand, $\|E_n' \pi_{\psi_{\alpha}}(b)E_n' - E_n' \pi_{\psi_{\alpha}}(b_n)E_n'\| < 1/n$, and $E_n' \pi_{\psi_{\alpha}}(b_n)E_n'$ is a compact operator on $E_n' \mathfrak{H}_{\psi_{\alpha}}$. Hence, there is a compact operator $T_\alpha$ on $\mathfrak{H}_\phi$ such that $\|U_\phi(b) - T_\alpha\| < 1/n$, because $E_n' \pi_{\psi_{\alpha}}(b)E_n' = \pi_{\psi_{\alpha}}(b)E_n'$. Therefore, $U_\phi(b)$ is a nonzero compact operator on $\mathfrak{H}_\phi$; this is a contradiction and so $\mathfrak{B} \cap \mathfrak{D} \subset \mathfrak{F}$.

Next, let us consider a $C^*$-algebra $\mathfrak{A}/\mathfrak{D}$, then $\mathfrak{B} + \mathfrak{D}/\mathfrak{D}$ is a $C^*$-subalgebra of $\mathfrak{A}/\mathfrak{D}$, because every *-homomorphic image of a $C^*$-algebra into another $C^*$-algebra is closed and the mapping $x \mapsto x + \mathfrak{D}(x \in \mathfrak{B})$ of $\mathfrak{B}$ into $\mathfrak{A}/\mathfrak{D}$ is *-homomorphic.

The state $\phi$ on $\mathfrak{B}$ can be canonically considered a pure state on $\mathfrak{B} + \mathfrak{D}/\mathfrak{D}$, because $\mathfrak{B} \cap \mathfrak{D} \subset \mathfrak{F}$ and the $C^*$-algebra $\mathfrak{B} + \mathfrak{D}/\mathfrak{D}$ is *-isomorphic to the $C^*$-algebra $\mathfrak{B}/\mathfrak{D}$. Take a pure state extension $\phi$ of $\phi$ to $\mathfrak{A}/\mathfrak{D}$, then we can define a pure state $\psi$ of $\mathfrak{A}$ by $\psi(y) = \phi(y + \mathfrak{D})$ for $y \in \mathfrak{A}$. Then we have $\psi = \phi$ on $\mathfrak{B}$ and so $\psi \in \mathcal{E}$.

Clearly $\mathfrak{D}(\psi_{\alpha}) \subset \mathfrak{D}(\chi)$; hence $\psi_{\alpha} \prec \psi$, and so by Zorn's lemma $\mathcal{E}$ contains a maximal element $\psi_0$.

Now we shall show $\mathfrak{D}(\psi_0) \cap \mathfrak{B} \subset \mathfrak{F}$. Assume that $\mathfrak{D}(\psi_0) \cap \mathfrak{B} \subset \mathfrak{F}$, then by the analogous discussion with the above, $\phi$ can be canonically considered a pure state on a $C^*$-subalgebra $\mathfrak{B} + \mathfrak{D}(\psi_0)/\mathfrak{D}(\psi_0)$ of $\mathfrak{A}/\mathfrak{D}(\psi_0)$; therefore we can have a pure state $\psi_\beta$ on $\mathfrak{A}$ such that $\psi_\beta(\mathfrak{D}(\psi_0)) = 0$ and $\psi_\beta = \phi$ on $\mathfrak{B}$; hence $\mathfrak{D}(\psi_\beta) \subset \mathfrak{D}(\psi_0)$, a contradiction.

On the other hand, $\mathfrak{D}(\psi_0) \cap \mathfrak{B} \subset \mathfrak{F}$ also implies a contradiction, be-
cause $\pi_{\psi_0}(b)$ is a compact operator on $\mathcal{D}_{\psi_0}$ for some $b \in (\mathcal{D}(\psi_0) \cap \mathcal{B})^c$; hence $\pi_{\psi_0}(b)E'$ is compact, where $E'$ is the orthogonal projection of $\mathcal{D}_{\psi_0}$ onto $[\pi_{\psi_0}(\mathcal{B})I_{\psi_0}]$; hence $U_\phi(b) = 0$ and so $b \in \mathcal{F}$.

Hence we can conclude that $\mathfrak{A}$ is of type I. This completes the proof.

REFERENCES


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