

ON ODD PERFECT NUMBERS. III

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It is not known whether or not odd perfect numbers exist. However, many interesting necessary conditions for an odd integer to be perfect have been found out. A bibliography of previous work on odd perfect numbers is given by McCarthy [3].

Throughout this paper n denotes an odd perfect number. The following results have been proved in [4], [6] and [5] respectively:

- (i) $\prod_{p|n} p/(p-1) < (175/96) \zeta(3) < 2.19125$.
- (ii) n is of the form $12t+1$ or $36t+9$.
- (iii) If n is of the form $36t+9$ and $5|n$, then

$$\sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \log\left(\frac{65}{61}\right) \quad (\sim 0.674).$$

- (iv) If n is of the form $36t+9$ and $5 \nmid n$, then

$$\sum_{p|n} \frac{1}{p} < \frac{1}{3} + \frac{1}{338} + \log\left(\frac{18}{13}\right) \quad (\sim 0.662).$$

The object of this paper is to improve the upper bound for the product $\prod_{p|n} p/(p-1)$ given by (i) above. We prove the following:

THEOREM. (α) *If n is of the form $12t+1$ and $5|n$,*

$$2 < \prod_{p|n} \frac{p}{p-1} < \frac{56791}{33612} \cdot \zeta(3) < 2.031002.$$

(β) *If n is of the form $12t+1$ and $5 \nmid n$,*

$$2 < \prod_{p|n} \frac{p}{p-1} < \frac{1760521}{1050375} \cdot \zeta(3) < 2.014754.$$

(γ) *If n is of the form $36t+9$ and $5|n$,*

$$2 < \prod_{p|n} \frac{p}{p-1} < \frac{318897}{177023} \cdot \zeta(3) < 2.165439.$$

(δ) *If n is of the form $36t+9$ and $5 \nmid n$,*

$$2 < \prod_{p|n} \frac{p}{p-1} < \frac{3706148208}{2125240975} \cdot \zeta(3) < 2.096234.$$

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PROOF. Euler proved that n must be of the form $p_0^{\alpha_0} \cdot x^2$, where p_0 is a prime of the form $4\lambda + 1$, α_0 is of the form $4\mu + 1$, $x > 1$ and $(p_0, x) = 1$. Hence we can write $n = p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where α_r is even for $1 \leq r \leq k$. We shall suppose without loss of generality that $p_1 < p_2 < \cdots < p_k$. Let $\sigma(n)$ denote the sum of all the positive divisors of n . Since n is a perfect number, we have $\sigma(n) = 2n$, from which it can easily be seen that

$$(A) \quad \prod_{r=0}^k \frac{p_r}{p_r - 1} = 2 \prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} > 2.$$

Throughout the following q_r denotes the r th prime, counting 2 as the first prime. We make use of the following well-known identity due to Euler:

$$(B) \quad \prod_{r=1}^{\infty} (1 - q_r^{-s})^{-1} = \zeta(s),$$

where $\zeta(s)$ is the Riemann Zeta function.

(a) Suppose n is of the form $12t + 1$. In this case, it has been proved in [4, p. 134] that p_0 is of the form $12N + 1$ and hence $p_0 \geq 13$.

(a₁) Suppose $5 | n$ and $7 | n$. Then $p_1 = 5$, $p_2 = 7$. Now $\alpha_2 \geq 4$. For, if $\alpha_2 = 2$, then $\sigma(p_2^{\alpha_2}) = 3.19$ and since $\sigma(n) = 2n$, it would follow that $3 | n$, which can not hold.

(a_{1.1}) If $p_0 = 13$, then $p_3 \geq 11$ and $p_r \geq q_{r+3}$ for $4 \leq r \leq k$. Since α_r is even for $1 \leq r \leq k$, $\alpha_2 \geq 4$ and $\alpha_0 \geq 1$, we have

$$\begin{aligned} \prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} &< (1 - 13^{-2})^{-1} (1 - 5^{-3})^{-1} (1 - 7^{-5})^{-1} (1 - 11^{-3})^{-1} \\ &\times \prod_{r=4}^k (1 - p_r^{-3})^{-1} \\ &< (1 - 13^{-2})^{-1} (1 - 5^{-3})^{-1} (1 - 7^{-5})^{-1} (1 - 11^{-3})^{-1} \\ &\times \prod_{r=7}^{\infty} (1 - q_r^{-3})^{-1} \\ &= \frac{(1 - 2^{-3})(1 - 3^{-3})(1 - 7^{-3})(1 - 13^{-3})}{(1 - 13^{-2})(1 - 7^{-5})} \cdot \zeta(3), \quad \text{by (B)} \\ &= \frac{56791}{67224} \cdot \zeta(3). \end{aligned}$$

(a_{1.2}) If $p_0 \neq 13$, then since p_0 is of the form $12N+1$, $p_0 \geq 37$. $p_r \geq q_{r+2}$ for $3 \leq r \leq k$. Hence

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 37^{-2})^{-1}(1 - 5^{-3})^{-1}(1 - 7^{-5})^{-1} \prod_{r=6}^{\infty} (1 - q_r^{-3})^{-1} < \frac{56791}{67224} \cdot \zeta(3).$$

Hence, by (A), (α) follows in the case (a₁).

(a₂) Suppose $5 \nmid n$ and $7 \nmid n$, then $p_1 = 5$ and $p_r \geq q_{r+3}$ for $2 \leq r \leq k$. Since α_0 is odd, $(1+p_0) \mid \sigma(p_0^{\alpha_0})$ and hence $\{(1+p_0)/2\} \mid n$, since $\sigma(n) = 2n$. Now, $p_0 \neq 13$. For, otherwise, by the above, it would follow that $7 \mid n$, which is not the case. Since p_0 is of the form $12N+1$, $p_0 \geq 37$. Hence

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 37^{-2})^{-1}(1 - 5^{-3})^{-1} \prod_{r=6}^{\infty} (1 - q_r^{-3})^{-1} < \frac{56791}{67224} \cdot \zeta(3).$$

Hence, by (A), (α) follows in this case also. Thus (α) is proved.

(a₃) Suppose $5 \nmid n$ and $7 \mid n$, then $p_1 = 7$. Now, $\alpha_1 \geq 4$. For, if $\alpha_1 = 2$, it would follow as in (a₁) that $3 \mid n$, which does not hold.

(a_{3.1}) If $p_0 = 13$, then $p_2 \geq 11$ and $p_r \geq q_{r+4}$ for $3 \leq r \leq k$. Hence

$$\begin{aligned} \prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} &< (1 - 13^{-2})^{-1}(1 - 7^{-5})^{-1}(1 - 11^{-3})^{-1} \prod_{r=7}^{\infty} (1 - q_r^{-3})^{-1} \\ &= \frac{(1 - 2^{-3})(1 - 3^{-3})(1 - 5^{-3})(1 - 7^{-3})(1 - 13^{-3})}{(1 - 13^{-2})(1 - 7^{-5})} \cdot \zeta(3) \\ &= \frac{1760521}{2100750} \cdot \zeta(3). \end{aligned}$$

(a_{3.2}) If $p_0 \neq 13$, then $p_0 \geq 37$ and $p_r \geq q_{r+3}$ for $2 \leq r \leq k$. Hence

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 37^{-2})^{-1}(1 - 7^{-5})^{-1} \prod_{r=6}^{\infty} (1 - q_r^{-3})^{-1} < \frac{1760521}{2100750} \cdot \zeta(3).$$

Hence, by (A), (β) follows in case (a₃).

(a₄) If $5 \nmid n$ and $7 \nmid n$, then $p_r \geq q_{r+4}$ for $1 \leq r \leq k$. As in (a₂), $p_0 \neq 13$ and hence $p_0 \geq 37$. Hence

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 37^{-2})^{-1} \prod_{r=5}^{\infty} (1 - q_r^{-3})^{-1} < \frac{1760521}{2100750} \cdot \zeta(3).$$

Hence, by (A), (β) follows in this case also. Thus (β) is proved.

(b) Suppose n is of the form $36t + 9$. Since $3 \mid n$, $p_1 = 3$.

(b₁) If $5 \mid n$, then $7 \nmid n$ in virtue of the result that $3 \cdot 5 \cdot 7$ does not divide n (proved by Kuhnel, p. 203 of [2]).

(b_{1.1}) Suppose $p_0 = 5$.

(b_{1.1.1}) If $11 \mid n$, then $\alpha_0 = 1$ in virtue of the result that $3 \cdot 5^2 \cdot 11$ does not divide n (proved by Kanold [1, p. 26]). In this case $p_2 = 11$ and $p_r \geq q_{r+3}$ for $3 \leq r \leq k$. Further, $\alpha_2 \geq 4$. For, if $\alpha_2 = 2$, then $\sigma(p_2^{\alpha_2}) = 133 = 7 \cdot 19$ and since $\sigma(n) = 2n$, it would follow that $7 \mid n$, which is not the case. Also, $\alpha_1 \geq 4$. For, if $\alpha_1 = 2$, then $\sigma(p_1^{\alpha_1}) = \sigma(3^2) = 13 \mid n$ and this implies that

$$\sum_{p \mid n} \frac{1}{p} > \frac{1}{3} + \frac{1}{5} + \frac{1}{11} + \frac{1}{13} > \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \log\left(\frac{65}{61}\right),$$

a contradiction to (iii). Hence

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 5^{-2})^{-1} (1 - 3^{-5})^{-1} (1 - 11^{-5})^{-1} \prod_{r=6}^{\infty} (1 - q_r^{-3})^{-1} < \frac{318897}{354046} \cdot \zeta(3).$$

Hence, by (A), (γ) follows in this case.

(b_{1.1.2}) Suppose $11 \nmid n$. Then $\alpha_1 \neq 4$. For, if $\alpha_1 = 4$, then $\sigma(p_1^{\alpha_1}) = 121 = 11^2$ and since $\sigma(n) = 2n$, it would follow that $11 \mid n$, which is not the case.

Hence, either $\alpha_1 = 2$ or $\alpha_1 \geq 6$.

Suppose $\alpha_1 = 2$. Then $\sigma(p_1^{\alpha_1}) = 13 \mid n$ and in this case both 17 and 19 together do not divide n . For, otherwise, it would follow that

$$\sum_{p \mid n} \frac{1}{p} > \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} > \frac{1}{3} + \frac{1}{5} + \frac{1}{13} + \log\left(\frac{65}{61}\right),$$

a contradiction to (iii). Hence

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 5^{-2})^{-1}(1 - 3^{-3})^{-1}(1 - 13^{-3})^{-1}(1 - 17^{-3})^{-1} \\ \times \prod_{r=9}^{\infty} (1 - q_r^{-3})^{-1},$$

if $17|n$, $19 \nmid n$; and

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 5^{-2})^{-1}(1 - 3^{-3})^{-1}(1 - 13^{-3})^{-1}(1 - 19^{-3})^{-1} \\ \times \prod_{r=9}^{\infty} (1 - q_r^{-3})^{-1},$$

if $17 \nmid n$, $19|n$; and

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 5^{-2})^{-1}(1 - 3^{-3})^{-1}(1 - 13^{-3})^{-1} \prod_{r=9}^{\infty} (1 - q_r^{-3})^{-1},$$

if $17 \nmid n$, $19 \nmid n$.

Since $1 < (1 - 19^{-3})^{-1} < (1 - 17^{-3})^{-1}$, it follows that in all the three cases, we have

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 5^{-2})^{-1}(1 - 3^{-3})^{-1}(1 - 13^{-3})^{-1}(1 - 17^{-3})^{-1} \\ \times \prod_{r=9}^{\infty} (1 - q_r^{-3})^{-1} \\ = \frac{318897}{354046} \cdot \zeta(3).$$

If $\alpha_1 \geq 6$, then

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 5^{-2})^{-1}(1 - 3^{-7})^{-1} \prod_{r=6}^{\infty} (1 - q_r^{-3})^{-1} \\ < \frac{318897}{354046} \cdot \zeta(3).$$

Hence, by (A), (γ) follows in the case $(b_{1.1.2})$ also.

$(b_{1.2})$ Suppose $p_0 \neq 5$. Then $p_2 = 5$. In this case, as in (a_2) , $p_0 \neq 13$. For, otherwise, it would follow that $7|n$, which can not hold. Since p_0 is of the form $4\lambda + 1$, $p_0 \geq 17$. $p_r \geq q_{r+2}$ for $3 \leq r \leq k$. Hence

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 17^{-2})^{-1}(1 - 3^{-3})^{-1}(1 - 5^{-3})^{-1} \prod_{r=5}^{\infty} (1 - q_r^{-3})^{-1} < \frac{318897}{354046} \cdot \zeta(3).$$

Hence, by (A), (γ) follows in this case also. Thus (γ) is proved.

(b₂) Suppose $5 \nmid n$. Since p_0 is of the form $4\lambda + 1$, $p_0 \geq 13$. Also, $p_0 \neq 29$. For, otherwise, it would follow as in (a₂) that $(1 + p_0)/2 = 3 \cdot 5 \mid n$, and this implies that $5 \mid n$, which is not the case.

(b_{2.1}) If $7 \mid n$, then $p_2 = 7$.

(b_{2.1.1}) Suppose $\alpha_1 = \alpha_2 = 2$. Then both 13 and 19 divide n , since $\sigma(p_1^{\alpha_1}) = 13$, $\sigma(p_2^{\alpha_2}) = 57 = 3 \cdot 19$ and $\sigma(n) = 2n$. In this case, neither 11 nor 17 divides n . For, otherwise, it would follow that

$$\sum_{p \mid n} \frac{1}{p} > \frac{1}{3} + \frac{1}{7} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} > \frac{1}{3} + \frac{1}{338} + \log\left(\frac{18}{13}\right),$$

a contradiction to (iv). Hence, either (1) $p_0 = 13$, $p_3 = 19$, $p_r \geq q_{r+6}$ for $4 \leq r \leq k$; or (2) $p_0 \neq 13$, $p_3 = 13$, $p_4 = 19$, $p_r \geq q_{r+4}$ for $5 \leq r \leq k$. In the second case, $p_0 \geq 37$, since p_0 is of the form $4\lambda + 1$, $p_0 \neq 13$, $17 \nmid n$ and $p_0 \neq 29$. In the first case,

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 13^{-2})^{-1}(1 - 3^{-3})^{-1}(1 - 7^{-3})^{-1}(1 - 19^{-3})^{-1} \times \prod_{r=9}^{\infty} (1 - q_r^{-3})^{-1} = \frac{1853074104}{2125240975} \cdot \zeta(3).$$

In the second case,

$$\prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} < (1 - 37^{-2})^{-1}(1 - 3^{-3})^{-1}(1 - 7^{-3})^{-1}(1 - 13^{-3})^{-1} \times (1 - 19^{-3})^{-1} \prod_{r=9}^{\infty} (1 - q_r^{-3})^{-1} < \frac{1853074104}{2125240975} \cdot \zeta(3).$$

(b_{2.1.2}) If at least one of α_1 and α_2 is not equal to 2, then either $\alpha_1 \geq 4$, $\alpha_2 \geq 4$ or $\alpha_1 \geq 4$, $\alpha_2 = 2$ or $\alpha_1 = 2$, $\alpha_2 \geq 4$. The proofs for the first two cases are omitted as they are similar to the previous proofs. In both these cases, we easily verify that the upper bound obtained for

$\prod_{p|n} p/(p-1)$ is less than the bound obtained in the third case. In the third case, we have

$$\begin{aligned} \prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} &< (1 - 13^{-2})^{-1} (1 - 3^{-3})^{-1} (1 - 7^{-6})^{-1} \prod_{r=5}^{\infty} (1 - q_r^{-3})^{-1} \\ &< \frac{1853074104}{2125240975} \cdot \zeta(3). \end{aligned}$$

Hence, by (A), (δ) follows in any case under ($b_{2.1}$).

($b_{2.2}$) If $7 \nmid n$, then $p_r \geq q_{r+3}$ for $2 \leq r \leq k$. Hence

$$\begin{aligned} \prod_{r=0}^k (1 - p_r^{-(\alpha_r+1)})^{-1} &< (1 - 3^{-3})^{-1} \prod_{r=5}^{\infty} (1 - q_r^{-3})^{-1} \\ &< \frac{1853074104}{2125240975} \cdot \zeta(3). \end{aligned}$$

Hence, by (A), (δ) follows in this case also. Thus (δ) is proved. Thus the proof of the theorem is complete.

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