SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

SOME THEOREMS OF BLOCH TYPE

P. S. CHIANG AND A. J. MACINTYRE

Very little is known about the constants in annular forms of Bloch's theorem [1], [5]. It seems interesting to extend the argument of Littlewood [4] connecting a Bloch type theorem with that of Landau and to use the numerical estimates due to Jenkins [3]. Of course we get only a weak form concerning values taken by a function and no information about an individual sheet of its Riemann surface. We quote Jenkins' form of Landau's theorem [3].

Let $g(z)=a_0+a_1z+\cdots$ be regular and never take the values 0 or 1 in $|z|<1$. Then $|a_1|\leqq 2|a_0|\{|\log|a_0||+5.94\}$. Using Littlewood's argument we then have

THEOREM 1. *Let $f(z)=z+a_2z^2+\cdots$ be regular in $|z|<1$. Then $f(z)$ takes all values on $|w|=r$ or all values on $|w|=2r$ if $r<1/35.64$.*

THEOREM 2. *Let $f(z)=z+a_2z^2+\cdots$ be regular in $|z|<1$. Then $f(z)$ takes all values in $r\leqq|w|\leqq 2r$ or all values in $4r\leqq|w|\leqq 8r$ if $r<1/118.8$.*

THEOREM 3. *Let $f(z)=z+a_2z^2+\cdots$ be regular in $|z|<1$ and let Δ_1 and Δ_2 be the disjoint circular domains $|w-\alpha_1|\leqq R_1$ and $|w-\alpha_2|\leqq R_2$. Then $f(z)$ takes all values in Δ_1 or in Δ_2 if*

(i) $|\alpha_1|+R_1\leqq|\alpha_2-\alpha_1|-R_1-R_2$,

(ii) $|\alpha_2-\alpha_1|+R_1+R_2<1/11.88$.

To prove Theorem 1 we suppose that there were missing values $re^{i\alpha}$ and $2re^{i\beta}$. Then

$$g(z) = (f(z) - re^{i\alpha})/(2re^{i\beta} - re^{i\alpha}) = a_0 + a_1z + \cdots.$$

Now $|g(0)|\leqq 1$ and we infer from Jenkins' result that

$$|g'(0)| = 1/|2re^{i\beta} - re^{i\alpha}| \leqq 2|a_0|(|\log|a_0|| + 5.94) \leqq 11.88.$$

It follows that $1/3r\leqq 11.88$ and that $r\geqq 1/35.64$.

To prove Theorem 2 we suppose that $f(z)$ has missing values $r_1e^{i\alpha}$ and $r_2e^{i\beta}$ with $r\leqq r_1\leqq 2r$ and $4r\leqq r_2\leqq 8r$.

Received by the editors April 11, 1967.

With

$$g(z) = (f(z) - r_1e^{i\alpha})/(r_2e^{i\beta} - r_1e^{i\alpha})$$

we again have $|g(0)| \leqq 1$ and hence

$$1/10r \leqq 1/|r_2e^{i\beta} - r_1e^{i\alpha}| \leqq 11.88.$$

To prove Theorem 3, suppose there are missing values w_1 and w_2 in Δ_1 and Δ_2 respectively.

With

$$g(z) = (f(z) - w_1)/(w_2 - w_1),$$

condition (i) ensures that $|g(0)| \leqq 1$ and hence

$$\operatorname*{Max}_{w_1 \in \Delta_1, w_2 \in \Delta_2} |w_2 - w_1| \geqq 1/11.88.$$

We should note that Hayman [2] has obtained the best possible constant (1/4) in Littlewood's Theorem. That is Hayman shows that $f(z) = z + a_2z^2 + \cdots$ takes all values on $|z| = r$ for some value of $r > 1/4$.

References

1. P. S. Chiang and A. J. Macintyre, *Upper bounds for a Bloch constant*, Proc. Amer. Math. Soc. **17** (1966), 26–31.

2. W. K. Hayman, *Some applications of the transfinite diameter to the theory of functions*, J. Analyse Math. **1** (1951), 155–179; Theorem IV, p. 167.

3. J. A. Jenkins, *On explicit bounds in Landau's theorem*, Canad. J. Math. **8** (1956), 423–425.

4. J. E. Littlewood, *Lectures on theory of functions*, Oxford Univ. Press, Oxford, 1944; pp. 225–226.

5. A. J. Macintyre, *On Bloch's theorem*, Math. Z. **44** (1938), 536–540.

Western Michigan University and
University of Cincinnati