THE HAHN-BANACH EXTENSION AND THE LEAST UPPER BOUND PROPERTIES ARE EQUIVALENT

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In [1] it was proved that a finite dimensional partially ordered vector space \( V \) has the Hahn-Banach extension property if and only if every nonempty set of elements in \( V \) bounded from above has a least upper bound. The methods used in [1] are used to prove that the least upper bound property and the extension property are equivalent. The assumption of finite dimensionality is dropped.

The terminology and results of [1] are assumed. Some of these will be repeated for easy reference. Let \((V, C)\) be a partially ordered vector space (OLS) with positive wedge \( C \), (i.e., \( V \) is a real linear space with a nonempty subset \( C \), such that \( C + C \subseteq C \), \( tC \subseteq C \), \( t \geq 0 \). The wedge \( C \) determines an order relation, \( u \preceq v \) if \( u - v \in C \), which is transitive and \( u \preceq v \) implies \( tu \preceq tv \) and \( u + w \preceq v + w \), \( t \geq 0 \), \( w \in V \). The wedge \( C \) is lineally closed if every line in \( V \) intersects \( C \) in a set which is closed relative to the line.

The OLS \((V, C)\) has the least upper bound property (LUBP) (or is a boundedly complete vector lattice) if every set of elements with an upper bound has a least upper bound (not necessarily unique). An OLS \((V, C)\) has the Hahn-Banach extension property (HBEP) if given (1) a real linear space \( Y \), (2) a linear subspace \( X \) of \( Y \), (3) a function \( p: Y \to V \) which is sublinear, (i.e., \( p(y) + p(y') \geq p(y + y') \) and \( p(ty) = tp(y) \), \( y, y' \in Y \), \( t \geq 0 \)) and (4) a linear function \( f: X \to V \) such that \( p(x) \geq f(x) \) for all \( x \in X \), then there is a linear extension \( F: Y \to V \) of \( f \) such that \( p(y) \geq F(y) \) for all \( y \in Y \).

It is proved in [2], [3] that \((V, C)\) has the LUBP if and only if \((V, C)\) has the HBEP and \( C \) is lineally closed. Thus the

**Theorem.** An OLS \((V, C)\) has the LUBP if and only if \((V, C)\) has the HBEP.

The theorem will be proved when it is shown that the HBEP for an OLS \((V, C)\) implies that \( C \) is lineally closed. This was done in [1] under the assumption that \( V \) was finite dimensional. In particular, in [1, Corollary 6.3] it is proved that a 2-dimensional OLS \((V, C)\) has the HBEP if and only if the wedge \( C \) is lineally closed.

It is clear that a wedge \( C \) in OLS \( V \) is lineally closed if and only if the

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intersection of \( C \) with any 2-dimensional subspace \( V_2 \) of \( V \) is lineally closed in \( V_2 \). Thus the theorem will be proved upon proof of the following lemma.

**Lemma.** If an OLS \((V, C)\) has the HBEP then every 2-dimensional ordered linear subspace \((V_2, C_2)\) of \((V, C)\) has the HBEP (where \( C_2 = V_2 \cap C \) is the positive wedge in the linear subspace \( V_2 \) of \( V \)).

**Proof of Lemma.** Assume that \((V_2, C_2)\) is an ordered linear subspace of \((V, C)\) which does not have the HBEP. That is \( C_2 \) is not lineally closed. Then it can be assumed [1, p. 219] that \( C_2 \) is one of the following four sets:

- \( C_2^{(1)} = \{ v \mid v \in V_2, v = ab_1 + bb_2, \text{ where } a, b \in \mathbb{R}, \text{ and both } a > 0, b > 0 \text{ or both } a = 0, b = 0 \} \), (the open first "quadrant" of \( V_2 \) plus the origin);
- \( C_2^{(2)} = \{ v \mid v \in V_2, v = ab_1 + bb_2, \text{ where } a, b \in \mathbb{R}, \text{ and } a \neq 0, b > 0, \text{ or } a = 0, b = 0 \} \), (the first "quadrant" of \( V_2 \) excluding the open bounding ray through \( b_1 \));
- \( C_2^{(3)} = \{ v \mid v \in V_2, v = ab_1 + bb_2, \text{ where } b > 0, \text{ or } a = 0, b = 0 \} \), (the open upper half plane plus the origin);
- \( C_2^{(4)} = \{ v \mid v \in V_2, v = ab_1 + bb_2, \text{ where } b > 0, \text{ or } a \geq 0, b = 0 \} \), (the open upper half plane plus the closed bounding ray through \( b_1 \)),

where \( R \) is the real number field and \( b_1, b_2 \) determine an appropriate basis for \( V_2 \).

Note that the wedge \( C_2^{(4)} \) is characterized by the property that \( v \in C_2^{(4)} \) if and only if \(-v \in C_2^{(4)}\) for every \( v \neq 0, v \) in \( V_2 \).

**Case 1.** Let \( C_2 = C_2^{(1)} \) or \( C_2^{(2)} \) or \( C_2^{(3)} \). We now refer to the specific example constructed in [1, p. 216, footnote (2)].

Let \( Y = \mathbb{R}^3, X = \{ (0, b, a) \mid b, a \in \mathbb{R} \} \). Define \( f_2: X \rightarrow \mathbb{R} \) by \( f_2(0, b, a) = a + b \) and \( f_1: X \rightarrow \mathbb{R} \) by \( f_1(0, b, a) = b \). Define \( p_2: Y \rightarrow V_2 \) by \( p_2(z) = p_2(0, b, a) = d + b, \forall z \in \mathbb{R} \).

Define:

\[
\begin{align*}
p_2(t, b, a) &= |a| + b + t, & t &\geq a, b \geq 0; \\
p_2(t, b, a) &= |a| + b + t + (a - t)^2/(a - t + b), & t &> a, b \geq 0; \\
p_2(t, b, a) &= |a| + t, & t &\geq a, b \geq 0; \\
p_2(t, b, a) &= |a| + a, & a &\geq t, b \leq 0.
\end{align*}
\]

Observe \( b_1 \in C_2 \), so \( b_1 \in C \).

Then by [1, p. 219, Case 1 and p. 220, Case 2(i)], \( p \) is a sublinear function from \( Y \) to \( V_2 \) with respect to the order determined by \( C_2 \).
If $f$ is linear from $X$ to $V_2$ and $p(x) - f(x) \in C_2$, $x \in X$. Further there is no linear extension $F$ of $f$ whose domain is all of $Y$, whose range is contained in $V_2$ and such that $p(y) - F(y) \in C_2$, $y \in Y$.

The assumption that $(V, C)$ has the HBEP guarantees that there is a linear extension $F$ of $f$ whose domain is $Y$ and whose range is a three dimensional subspace $V_3$ of $V$ which properly contains $V_2$ and such that $p(y) - F(y) \in C_3$, $y \in Y$, where $C_3 = V_3 \cap C$. Consider $-F(1, 0, 0) = b_3$. Then $b_3 \neq 0$ and $\{b_1, b_2, b_3\}$ is a basis for $V_3$. Also,

(i) $p(t, b, a) - F(t, b, a) = (|a| + t)b_1 + (|a| - a + t)b_2 + tb_3 \in C_3$, $t \geq a$, $b \geq 0$;

(ii) $p(t, b, a) - F(t, b, a) = (|a| + t + (a - t)^2/(a - t + b))b_1 + (|a| - a + t + (a - t)^2/(a - t + b))b_2 + tb_3 \in C_3$, $a > t$, $b \geq 0$.

(a) Consider $a = 0$, $b = 0$ in (i). Then (i) implies that $t(b_1 + b_2 + b_3) \in C_3$, $t \geq 0$.

(b) Consider $a = 0$, $b = t^2 + t$, $t \leq -1$ in (ii). Then (ii) implies that $ub_1 + ub_2 + (u - 1)b_3 \in C_3$, $u \leq 0$.

Using the properties of a wedge, statements (a) and (b) imply that the line $L = \{t(b_1 + b_2 + b_3) \ | \ t \in \mathbb{R}\} \subset \overline{C_3} \subset \overline{C}$ (where $\overline{C}$ is the lineal closure of $C$) with the $\frac{1}{2}$-line for $t \geq 0$ in $C$ (by (a)) and the open $\frac{1}{2}$-line for $t < 0$ in $\overline{C}$ but possibly not in $C$: For letting $u$ approach $-\infty$ in (b), the resulting rays through $u(b_1 + b_2 + b_3) - b_3$ approach the ray through $-(b_1 + b_2 + b_3)$ and hence $-(b_1 + b_2 + b_3) \in \overline{C}$.

If $L \subset C_3$, taking $a \geq 0$ in (i), it follows that $ab_1 + t(b_1 + b_2 + b_3) \in C$ for $t \geq a$. Hence, by the wedge properties of $C_3$, $b_1 \in C$, a contradiction.

Therefore, it must be assumed that $L^+ = \{t(b_1 + b_2 + b_3) \ | \ t \geq 0\} \subset C$ but $L^- = \{t(b_1 + b_2 + b_3) \ | \ t < 0\} \subset C$. Considering the subspace spanned by $L$ and $b_1 + b_2$, one obtains an induced wedge in this subspace of the type $C_2^{(4)}$. The next case considers the possibility of a wedge of this type.

Case 2. Assume $C_2$ has the form $C_2^{(4)}$. Then referring to [1, p. 221, Case (2v) and p. 217, Example 2] there is an example of a $C_2$-sublinear (hence $C$-sublinear) function $q: R_2 \rightarrow V_2$ and a linear function $f: X \rightarrow V_2$, where $X = \{(0, a) \ | \ a \in \mathbb{R}\}$, which is $C_2$-dominated by $q$ and which has no linear extension $F$ with domain $R_2$ and range $V_2$ which is $C_2$-dominated by $q$. Specifically,

\[ f(0, a) = ab_2, \quad a \in R; \]
\[ q(y) = r_1(y)b_1 + r_2(y)b_2, \quad y \in R_2, \]

where

\[ r_1(t, a) = - (at)^{1/2}, \quad t \geq 0 \quad \text{and} \quad a \geq 0, \]
\[ r_1(t, a) = 0, \quad t \leq 0 \quad \text{or} \quad a \leq 0, \]
\[ r_2(t, a) = \begin{cases} \left\lfloor a \right\rfloor + t, & t \geq 0, \\ a + at/(a-t), & t < 0 \text{ and } a > 0, \\ -a, & t \leq 0 \text{ and } a \leq 0. \end{cases} \]

The assumption that \((V, C)\) has the HBEP implies that there exists a linear extension \(F\) of \(f\) with domain \(R_2\) and range in a linear subspace \(V_3\) of \(V\) with induced wedge \(C_3 = C \cap V_3\) where \(V_3 \supseteq V_2\) and \(F\) is \(C_3\)-dominated by \(q\). As in Case 1, set \(-F(1, 0) = b_3\). Then,

(i) \(q(t, a) - F(t, a) = -(at)^{1/2}b_1 + tb_2 + tb_3, t \geq 0, a \geq 0;\)

(iv) \(q(t, a) - F(t, a) = (at/(a-t))b_2 + tb_3, t < 0, a > 0.\)

(a) In (iv) set \(a = -dt, d > 0\), then \((d/(d+1))tb_2 + tb_3 \in C, t < 0.\)

(b) In (i), set \(a = k^2/t\). Then \(-kb_1 + t(b_2 + b_3) \in C, t > 0, k \geq 0.\)

Let

\[ Q_3 = \{ \alpha b_1 + \beta (b_2 + b_3) + \gamma b_2 \mid \gamma > 0, \text{ or } \gamma = 0, \beta > 0, \text{ or } \gamma = 0, \beta = 0, \alpha \geq 0 \}. \]

Clearly, \(Q_3\) is a wedge. The wedge \(C_3 \supseteq Q_3\) for if \(\gamma > 0,\)

(1) \(ab_1 + \gamma b_2 \in C_2 \subseteq C_3\) for all \(a,\)

and by (b),

(2) \(-kb_1 + t(b_2 + b_3) \in C_3, t > 0, k \geq 0.\)

Adding (1) and (2),

(3) \(\alpha b_1 + t(b_2 + b_3) + \gamma b_2 \in C_3\)

where \((a-k)=\alpha\) is arbitrary, and \(t > 0.\) By (a),

(4) \(\beta b_2 + ((d + 1)/d)\beta b_3 \in C_3, d > 0, \beta < 0.\)

Adding (1) and (4),

(5) \(ab_1 + \beta (b_2 + b_3) + ((d + 1)/d - 1)\beta + \gamma)b_3 \in C_3\)

where \(\gamma > 0, \beta < 0, a\) is arbitrary, and \(d > 0.\) Consider any number \(\gamma' > 0, d\) large enough and \(\gamma > 0\) so that \((d/(d+1)) - 1)\beta + \gamma = \gamma'.\) Therefore,

(6) \(ab_1 + \beta (b_2 + b_3) + \gamma' b_3 \in C_3, \) for \(\gamma' > 0, \beta < 0, a\) arbitrary.

If \(\beta = 0,\) then

(7) \(ab_1 + \gamma b_2 \in C_2 \subseteq C_3\) for all \(\gamma > 0, \text{ and all } \alpha.\)

Combining statements (3), (6) and (7),

(8) \(\alpha b_1 + \beta (b_2 + b_3) + \gamma b_2 \subseteq C, \gamma > 0.\)
If $\gamma = 0$ and $\beta > 0$, then

\[(9) \quad ab_1 \in C_2 \subset C_3, \quad \text{for } a \geq 0.\]

By (b),

\[(10) \quad -kb_1 + \beta(b_2 + b_3) \in C_3, \quad \beta > 0, \quad k > 0.\]

Adding (9) and (10),

\[(11) \quad \alpha b_1 + \beta(b_2 + b_3) \in C_3,\]

for arbitrary $\alpha = a - k$, $\beta > 0$.

If $\gamma = 0$, $\beta = 0$ and $\alpha b_1 \in C_3$ then $\alpha b_1 \in C_2$ and thus $\alpha \geq 0$. This statement plus (8) and (11) show that $Q_3 \subset C_3$.

It may be assumed that $Q_3 = C_3$. For otherwise it follows that $-b_1 \in C_3 \subset C$, a contradiction of the initial assumption in Case 2. To prove this, observe that the wedge $Q_3$ is characterized by the property that $v \in Q_3$ if and only if $-v \in Q_3$, for every $v \in V_3$, $v \neq 0$. Equivalently, $Q_3$ consists of the open half-space of $V_3$ containing $b_2$ bounded by the subspace $V'_2$ spanned by $b_1$ and $b_2 + b_3$, joined with the open half-space in $V'_2$ containing $b_2 + b_3$ and bounded by the 1-dimensional subspace containing $b_1$, to which is adjoined the closed half-ray through $b_1$. If $v$ is an element in $C_3$ but not in $Q_3$, it then follows that $C_3$ either contains the closed half-space containing $Q_3$ (if $v \in V'_2$) or $C_3 = V_3$ (if $v \notin V'_2$). In both cases $-b_1 \in C_3$, the contradiction.

Let $(W, K)$ be an OLS where $K$ is a set such that every 2-dimensional linear subspace of $W$ intersects $K$ in a wedge of type $C_2^{(4)}$. The set $K$ is a wedge, for if $v_1$ and $v_2$ are in $K$ and $V'_2$ is a 2-dimensional linear subspace containing $v_1$ and $v_2$ then $V'_2$ cuts $K$ in a wedge (of type $C_2^{(4)}$) and so $v_1 + v_2$ and $\lambda v_1$, $\lambda \geq 0$, are in $V'_2 \cap K \subset K$.

Further, $K$ must be a half-space. For if $v \in W$, let $V'_2$ be a 2-dimensional subspace of $W$ containing $v$. Then $V'_2$ cuts $K$ in a wedge of type $C_2^{(4)}$. Since such a wedge is characterized by the property that every nonzero vector (in its plane) or its negative, but not both, is in the wedge, it follows that $K$ has this property also. Since $K$ is convex, $K$ is a half-space.

The wedge $(V_3, C_3)$ (above) is an OLS of the same form as $(W, K)$. Additionally it is clear that the OLS $(W', K')$ where $W'$ is the subspace bounding $K$ and $K' = W' \cap K$ inherits the property that every 2-dimensional linear subspace of $W'$ intersects $K'$ in a wedge of type $C_2^{(4)}$.

The OLS $(W, K)$ does not have the HBEP. For if $B_2$ is an element of $K$ which is not in $W'$ and $B_1$ is an element of $K'$ which is not in the
hyperplane bounding $K'$ in $W'$, and $r_1, r_2, f_1, f_2$ are defined as in the beginning of Case 2, then

$q: R_2 \to W, \quad$ where $\quad q(y) = r_2(y)B_2 + r_1(y)B_1, \quad y \in R_2, \quad$ is $K$-sublinear;

$f: X \to W, \quad$ where $\quad f(0, a) = aB_2, \quad a \in R, \quad$ is linear and

$$q(x) - f(x) \in K, \quad x \in X.$$ 

But there is no linear extension $F$ of $f$ with domain $R_2$ and range $W$ which is $K$-dominated by $q$. If there were such an extension $F$, the union of the ranges of $q$ and of $F$ would span a 3-dimensional subspace with basis $B_2, B_1, B_3$ where $B_3$ can be taken to be in the subspace $W''$ bounding $K'$ in $W'$. Note that if $w = a_2B_2 + a_1B_1 + w''$ with $w''$ in $W''$ and $w$ in $K$, then $a_2 > 0$ or if $a_2 = 0$ then $a_1 \leq 0$.

Thus if $F_i$ and $q_i$ ($i = 1, 2, 3$) are the coordinate functions of $F$ and $q$ respectively, then the fact that

$$q(t, a) - F(t, a) = (q_1(t, a) - F_1(t, a))B_2 + (q_2(t, a) - F_2(t, a))B_1 - F_3(t, a)B_3$$

is in $K$ implies that $(q_2(t, a) - F_2(t, a))B_2 + (q_1(t, a) - F_1(t, a))B_1$ is also in $K$ for all $(t, a)$. Hence $F'$, where $F'(t, a) = F_2(t, a)B_2 + F_1(t, a)B_1$, would be an extension of $f$, a contradiction of the fact that $f$ has no extension which is $C_3$-dominated by $q$ where $C_3$ is the induced wedge of type $C_2^3$ in the subspace $V_2$ spanned by $B_1$ and $B_2$.

If $(W, K)$ is an ordered linear subspace of $(V, C)$ then, since $(V, C)$ is assumed to have the HBEP, a linear extension $F$ of $f$ will exist with range contained in $V$ which is $C$-dominated by $q$. If $W_3$ is the subspace spanned by the union of the ranges of $F$ and $q$ with induced wedge $K_3 = W_3 \cap K$, it follows that $(W_3, K_3)$ can be identified with $(V_3, C_3)$ considered previously (with $C_3 = Q_3$) upon identifying $B_1$ with $b_1$, $B_2$ with $b_2$ and $B_{1,5}$ with $b_2 + b_3$ so that

$$K_3 = \{ v \mid v = 0 \text{ or } v = a_1B_1 + a_{1,5}B_{1,5} + a_2B_2 \text{ where } a_2 > 0, \text{ or}$$

$$a_2 = 0, a_{1,5} > 0, \text{ or } a_2 = 0, a_{1,5} = 0, a_1 > 0 \}.$$ 

The ordered linear subspace $(\tilde{W}, \tilde{K})$ of $V$ spanned by $W$ and $B_{1,5}$ has induced wedge $\tilde{K} = \tilde{W} \cap C$ of the same type as $K$. This will be proved when it is shown that if $\tilde{w} \in \tilde{W}$ and $\tilde{w} \neq 0$, then $\tilde{w} \in \tilde{K}$ if and only if $-\tilde{w} \in \tilde{K}$. Let $\tilde{w} = \alpha B_2 + \beta B_{1,5} + \gamma B_1 + w''$ where $w'' \in W''$, the subspace bounding $K'$, where $K'$ is the wedge in $W'$, the subspace bounding $K$. If $\alpha > 0$, then $\tilde{w} = u_1 + u_2$, where $u_1 = (\alpha/2)B_2 + \beta B_{1,5} \in K_3$ and $u_2 = (\alpha/2)B_2 + \gamma B_1 + w'' \in K$. Therefore $\tilde{w} \in K_3 + K \subset \tilde{K}$. If $\alpha < 0$, and $\tilde{w} \in K$, then from the previous sentence, $u = -\alpha B_2 - \beta B_{1,5}$
\(- (\gamma + 1)B_1 - w'' \in \bar{K}\). Hence \(u + \bar{w} = -B_1 \in \bar{K} \cap W = K\), a contradiction, since \(B_1 \in K\).

If \(\alpha = 0, \beta > 0\), then
\[
\bar{w} = (\beta B_{1.5} + (\gamma - 1)B_1) + (B_1 + w'') \in K_{.5} + K \subseteq \bar{K}.
\]

If \(\alpha = 0, \beta < 0\) and \(\bar{w} \in \bar{K}\), then from the previous sentence \(u = -\beta B_{1.5} - (\gamma + 1)B_1 - w'' \in \bar{K}\). Therefore, \(u + \bar{w} = -B_1 \in \bar{K} \cap W = K\), again a contradiction.

If \(\alpha = 0, \beta = 0\), then \(\bar{w} \in W\). Hence, if \(\bar{w} \neq 0, \bar{w} \in K\) if and only if \(-\bar{w} \notin K\). Thus \(\bar{K}\) has the form asserted.

A Zorn's Lemma argument guarantees the existence of a maximal ordered linear subspace \((W^*, K^*)\) of \((V, C)\) whose induced wedge \(K^*\) is of the same form as \(K\) and which does not have the HBEP. The previous argument proves that \((W^*, K^*) = (V, C)\), a contradiction. Hence, no 2-dimensional cut of \(C\) by a subspace is of the type \(C_s^{(4)}\). Thus, every 2-dimensional cut of \(C\) is closed and the Lemma and the Theorem are proved.

**Bibliography**


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