THE HAHN-BANACH EXTENSION AND THE LEAST UPPER BOUND PROPERTIES ARE EQUIVALENT

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In [1] it was proved that a finite dimensional partially ordered vector space V has the Hahn-Banach extension property if and only if every nonempty set of elements in V bounded from above has a least upper bound. The methods used in [1] are used to prove that the least upper bound property and the extension property are equivalent. The assumption of finite dimensionality is dropped.

The terminology and results of [1] are assumed. Some of these will be repeated for easy reference. Let (V, C) be a *partially ordered vector space* (OLS) with *positive wedge* C, (i.e., V is a real linear space with a nonempty subset C, such that $C+C\subset C$, $tC\subset C$, $t\geqq 0$. The wedge C determines an order relation, $u\geqq v$ if $u-v\in C$, which is transitive and $u\geqq v$ implies $tu\geqq tv$ and $u+w\geqq v+w$, $t\geqq 0$, $w\in V$). The wedge C is *lineally closed* if every line in V intersects C in a set which is closed relative to the line.

The OLS (V, C) has the *least upper bound property* (LUBP) (or is a *boundedly complete vector lattice*) if every set of elements with an upper bound has a least upper bound (not necessarily unique). An OLS (V, C) has the *Hahn-Banach extension property* (HBEP) if given (1) a real linear space Y, (2) a linear subspace X of Y, (3) a function p: $Y\to V$ which is sublinear, (i.e., $p(y)+p(y')\geqq p(y+y')$ and $p(ty)=tp(y)$, y, $y'\in Y$, $t\geqq 0$) and (4) a linear function f: $X\to V$ such that $p(x)\geqq f(x)$ for all $x\in X$, then there is a linear extension F: $Y\to V$ of f such that $p(y)\geqq F(y)$ for all $y\in Y$.

It is proved in [2], [3] that *(V, C) has the LUBP if and only if (V, C) has the HBEP and C is lineally closed.* Thus the

THEOREM. *An OLS (V, C) has the LUBP if and only if (V, C) has the HBEP.*

The theorem will be proved when it is shown that the HBEP for an OLS (V, C) implies that C is lineally closed. This was done in [1] under the assumption that V was finite dimensional. In particular, in [1, Corollary 6.3] it is proved that a 2-dimensional OLS (V, C) has the HBEP if and only if the wedge C is lineally closed.

It is clear that *a wedge C in OLS V is lineally closed if and only if the*

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intersection of C with any 2-dimensional subspace V_2 of V is lineally closed in V_2. Thus the theorem will be proved upon proof of the following lemma.

LEMMA. *If an* OLS (V, C) *has the* HBEP *then every 2-dimensional ordered linear subspace* (V_2, C_2) *of* (V, C) *has the* HBEP (*where* $C_2 = V_2 \cap C$ *is the positive wedge in the linear subspace* V_2 *of* V).

PROOF OF LEMMA. Assume that (V_2, C_2) is an ordered linear subspace of (V, C) which does not have the HBEP. That is C_2 is not lineally closed. Then it can be assumed [**1**, p. 219] that C_2 is one of the following four sets:

$C_2^{(1)} = \{v \mid v \in V_2,\ v = ab_1 + bb_2$, where $a, b \in R$, and both $a > 0$, $b > 0$ or both $a = 0$, $b = 0\}$, (the open first "quadrant" of V_2 plus the origin);

$C_2^{(2)} = \{v \mid v \in V_2,\ v = ab_1 + bb_2$, where $a, b \in R$, and $a \geqq 0$, $b > 0$, or $a = 0$, $b = 0\}$, (the first "quadrant" of V_2 excluding the open bounding ray through b_1);

$C_2^{(3)} = \{v \mid v \in V_2,\ v = ab_1 + bb_2$, where $b > 0$, or $a = 0$, $b = 0\}$, (the open upper half plane plus the origin);

$C_2^{(4)} = \{v \mid v \in V_2,\ v = ab_1 + bb_2$, where $b > 0$, or $a \geqq 0$, $b = 0\}$, (the open upper half plane plus the closed bounding ray through b_1), where R is the real number field and b_1, b_2 determine an appropriate basis for V_2.

Note that the wedge $C_2^{(4)}$ is characterized by the property that $v \in C_2^{(4)}$ if and only if $-v \notin C_2^{(4)}$ for every $v \neq 0$, v in V_2.

Case 1. Let $C_2 = C_2^{(1)}$ or $C_2^{(2)}$ or $C_2^{(3)}$. We now refer to the specific example constructed in [**1**, p. 216, footnote (2)].

Let $Y = R_3$, $X = \{(0, b, a) \mid b, a \in R\}$. Define $f_2 \colon X \to R$ by $f_2(0, b, a) = a + b$ and $f_1 \colon X \to R$ by $f_1(0, b, a) = b$. Define $p_2 \colon Y \to R$ by

$$p_2(t, b, a) = |a| + b + t, \qquad t \geqq a,\ b \geqq 0;$$

$$p_2(t, b, a) = |a| + b + t + (a - t)^2/(a - t + b), \qquad a > t,\ b \geqq 0;$$

$$p_2(t, b, a) = |a| + t, \qquad t \geqq a,\ b \leqq 0;$$

$$p_2(t, b, a) = |a| + a, \qquad a \geqq t,\ b \leqq 0.$$

Define:

$$p \colon Y \to V_2,\ p(z) = p_2(z)(b_1 + b_2), \qquad z \in Y;$$

$$f \colon X \to V_2,\ f(x) = f_1(x)b_1 + f_2(x)b_2, \qquad x \in X.$$

Observe $b_1 \notin C_2$, so $b_1 \notin C$.

Then by [**1**, p. 219, Case 1 and p. 220, Case 2(i)], p is a sublinear function from Y to V_2 with respect to the order determined by C_2,

f is linear from X to V_2 and $p(x)-f(x)\in C_2$, $x\in X$. Further there is no linear extension F of f whose domain is all of Y, whose range is contained in V_2 and such that $p(y)-F(y)\in C_2$, $y\in Y$.

The assumption that (V, C) has the HBEP guarantees that there is a linear extension F of f whose domain is Y and whose range is a three dimensional subspace V_3 of V which properly contains V_2 and such that $p(y)-F(y)\in C_3$, $y\in Y$, where $C_3=V_3\cap C$. Consider $-F(1, 0, 0)=b_3$. Then $b_3\neq 0$ and $\{b_1, b_2, b_3\}$ is a basis for V_3. Also,

(i) $p(t, b, a)-F(t, b, a)=(|a|+t)b_1+(|a|-a+t)b_2+tb_3\in C_3$, $t\geqq a$, $b\geqq 0$;

(ii) $p(t,b,a)-F(t,b,a)=(|a|+t+(a-t)^2/(a-t+b))b_1+(|a|-a+t+(a-t)^2/(a-t+b))b_2+tb_3\in C_3$, $a>t$, $b\geqq 0$.

(a) Consider $a=0$, $b=0$ in (i). Then (i) implies that $t(b_1+b_2+b_3)\in C_3$, $t\geqq 0$.

(b) Consider $a=0$, $b=t^2+t$, $t\leqq -1$ in (ii). Then (ii) implies that $ub_1+ub_2+(u-1)b_3\in C_3$, $u\leqq 0$.

Using the properties of a wedge, statements (a) and (b) imply that the line $L=\{t(b_1+b_2+b_3)\,|\,t\in R\}\subset\overline{C}_3\subset\overline{C}$ (where $\overline{C}$ is the lineal closure of C) with the $\frac{1}{2}$-line for $t\geqq 0$ in C (by (a)) and the open $\frac{1}{2}$-line for $t<0$ in $\overline{C}$ but possibly not in C: For letting u approach $-\infty$ in (b), the resulting rays through $u(b_1+b_2+b_3)-b_3$ approach the ray through $-(b_1+b_2+b_3)$ and hence $-(b_1+b_2+b_3)\in\overline{C}_3$.

If $L\subset C_3$, taking $a\geqq 0$ in (i), it follows that $ab_1+t(b_1+b_2+b_3)\in C$ for $t\geqq a$. Hence, by the wedge properties of C_3, $b_1\in C$, a contradiction.

Therefore, it must be assumed that $L^+=\{t(b_1+b_2+b_3)\,|\,t\geqq 0\}\subset C$ but $L^-=\{t(b_1+b_2+b_3)\,|\,t<0\}\not\subset C$. Considering the subspace spanned by L and b_1+b_2, one obtains an induced wedge in this subspace of the type $C_2^{(4)}$. The next case considers the possibility of a wedge of this type.

Case 2. Assume C_2 has the form $C_2^{(4)}$. Then referring to [**1**, p. 221, Case (2v) and p. 217, Example 2] there is an example of a C_2-sublinear (hence C-sublinear) function $q: R_2\to V_2$ and a linear function $f: X\to V_2$, where $X=\{(0, a)\,|\,a\in R\}$, which is C_2-dominated by q and which has no linear extension F with domain R_2 and range V_2 which is C_2-dominated by q. Specifically,

$$f(0, a) = ab_2, \qquad a\in R;$$

$$q(y) = r_1(y)b_1 + r_2(y)b_2, \qquad y\in R_2,$$

where

$$r_1(t, a) = -(at)^{1/2}, \qquad t\geqq 0 \text{ and } a\geqq 0,$$

$$r_1(t, a) = 0, \qquad t\leqq 0 \text{ or } a\leqq 0,$$

$$r_2(t, a) = |a| + t, \qquad t \geqq 0,$$
$$r_2(t, a) = a + at/(a - t), \qquad t < 0 \quad \text{and} \quad a > 0,$$
$$r_2(t, a) = -a, \qquad t \leqq 0 \quad \text{and} \quad a \leqq 0.$$

The assumption that (V, C) has the HBEP implies that there exists a linear extension F of f with domain R_2 and range in a linear subspace V_3 of V with induced wedge $C_3 = C \cap V_3$ where $V_3 \supsetneqq V_2$ and F is C_3-dominated by q. As in Case 1, set $-F(1, 0) = b_3$. Then,

(i) $q(t, a) - F(t, a) = -(at)^{1/2}b_1 + tb_2 + tb_3$, $t \geqq 0$, $a \geqq 0$;

(iv) $q(t, a) - F(t, a) = (at/(a-t))b_2 + tb_3$, $t < 0$, $a > 0$.

(a) In (iv) set $a = -dt$, $d > 0$, then $(d/(d+1))tb_2 + tb_3 \in C$, $t < 0$.

(b) In (i), set $a = k^2/t$. Then $-kb_1 + t(b_2 + b_3) \in C$, $t > 0$, $k \geqq 0$.

Let

$$Q_3 = \{\alpha b_1 + \beta(b_2 + b_3) + \gamma b_2 \mid \gamma > 0, \text{ or } \gamma = 0,\ \beta > 0, \text{ or } \gamma = 0,\ \beta = 0,\ \alpha \geqq 0\}.$$

Clearly, Q_3 is a wedge. The wedge $C_3 \supset Q_3$ for if $\gamma > 0$,

$$(1) \qquad ab_1 + \gamma b_2 \in C_2 \subset C_3 \quad \text{for all } a,$$

and by (b),

$$(2) \qquad -kb_1 + t(b_2 + b_3) \in C_3, \qquad t > 0,\ k \geqq 0.$$

Adding (1) and (2),

$$(3) \qquad \alpha b_1 + t(b_2 + b_3) + \gamma b_2 \in C_3$$

where $(a - k) = \alpha$ is arbitrary, and $t > 0$. By (a),

$$(4) \qquad \beta b_2 + ((d + 1)/d)\beta b_3 \in C_3, \qquad d > 0, \beta < 0.$$

Adding (1) and (4),

$$(5) \qquad ab_1 + \beta(b_2 + b_3) + (((d + 1)/d - 1)\beta + \gamma)b_3 \in C_3$$

where $\gamma > 0$, $\beta < 0$, a is arbitrary, and $d > 0$. Consider any number $\gamma' > 0$, d large enough and $\gamma > 0$ so that $(d/(d+1) - 1)\beta + \gamma = \gamma'$. Therefore,

$$(6) \qquad ab_1 + \beta(b_2 + b_3) + \gamma' b_3 \in C_3, \text{ for } \gamma' > 0,\ \beta < 0,\ a \text{ arbitrary}.$$

If $\beta = 0$, then

$$(7) \qquad \alpha b_1 + \gamma b_2 \in C_2 \subset C_3 \quad \text{for all } \gamma > 0, \quad \text{and all } \alpha.$$

Combining statements (3), (6) and (7),

$$(8) \qquad \alpha b_1 + \beta(b_2 + b_3) + \gamma b_2 \in C, \qquad \gamma > 0.$$

If $\gamma=0$ and $\beta>0$, then

$$ab_1 \in C_2 \subset C_3, \quad \text{for } a \geqq 0. \tag{9}$$

By (b),

$$-kb_1 + \beta(b_2 + b_3) \in C_3, \qquad \beta > 0,\ k > 0. \tag{10}$$

Adding (9) and (10),

$$\alpha b_1 + \beta(b_2 + b_3) \in C_3, \tag{11}$$

for arbitrary $\alpha = a - k$, $\beta > 0$.

If $\gamma=0$, $\beta=0$ and $\alpha b_1 \in C_3$ then $\alpha b_1 \in C_2$ and thus $\alpha \geqq 0$. This statement plus (8) and (11) show that $Q_3 \subset C_3$.

It may be assumed that $Q_3 = C_3$. For otherwise it follows that $-b_1 \in C_3 \subset C$, a contradiction of the initial assumption in Case 2. To prove this, observe that the wedge Q_3 is characterized by the property that $v \in Q_3$ if and only if $-v \notin Q_3$, for every $v \in V_3$, $v \neq 0$. Equivalently, Q_3 consists of the open half-space of V_3 containing b_2 bounded by the subspace V_2' spanned by b_1 and b_2+b_3, joined with the open half-space in V_2' containing b_2+b_3 and bounded by the 1-dimensional subspace containing b_1, to which is adjoined the closed half-ray through b_1. If v is an element in C_3 but not in Q_3, it then follows that C_3 either contains the closed half-space containing Q_3 (if $v \in V_2'$) or $C_3 = V_3$ (if $v \notin V_2'$). In both cases $-b_1 \in C_3$, the contradiction.

Let (W, K) be an OLS where K is a set such that every 2-dimensional linear subspace of W intersects K in a wedge of type $C_2^{(4)}$. The set K is a wedge, for if v_1 and v_2 are in K and V_2' is a 2-dimensional linear subspace containing v_1 and v_2 then V_2' cuts K in a wedge (of type $C_2^{(4)}$) and so v_1+v_2 and λv_1, $\lambda \geqq 0$, are in $V_2' \cap K \subset K$.

Further, K must be a half-space. For if $v \in W$, let V_2' be a 2-dimensional subspace of W containing v. Then V_2' cuts K in a wedge of type $C_2^{(4)}$. Since such a wedge is characterized by the property that every nonzero vector (in its plane) or its negative, but not both, is in the wedge, it follows that K has this property also. Since K is convex, K is a half-space.

The wedge (V_3, C_3) (above) is an OLS of the same form as (W, K). Additionally it is clear that the OLS (W', K') where W' is the subspace bounding K and $K' = W' \cap K$ inherits the property that every 2-dimensional linear subspace of W' intersects K' in a wedge of type $C_2^{(4)}$.

The OLS (W, K) *does not have the* HBEP. For if B_2 is an element of K which is not in W' and B_1 is an element of K' which is not in the hyperplane bounding K' in W', and r_1, r_2, f_1, f_2 are defined as in the beginning of Case 2, then

$$q\colon R_2 \to W, \quad \text{where} \quad q(y) = r_2(y)B_2 + r_1(y)B_1, y \in R_2, \text{is } K\text{-sublinear};$$
$$f\colon X \to W, \quad \text{where} \quad f(0, a) = aB_2, \ a \in R, \ \text{is linear and}$$
$$q(x) - f(x) \in K, \ x \in X.$$

But there is no linear extension F of f with domain R_2 and range W which is K-dominated by q. If there were such an extension F, the union of the ranges of q and of F would span a 3-dimensional subspace with basis B_2, B_1, B_3 where B_3 can be taken to be in the subspace W'' bounding K' in W'. Note that if $w = a_2B_2 + a_1B_1 + w''$ with w'' in W'' and w in K, then $a_2 > 0$ or if $a_2 = 0$ then $a_1 \geqq 0$.

Thus if F_i and q_i $(i = 1, 2, 3)$ are the coordinate functions of F and q respectively, then the fact that

$$q(t, a) - F(t, a) = (q_2(t, a) - F_2(t, a))B_2$$
$$+ (q_1(t, a) - F_1(t, a))B_1 - F_3(t, a)B_3$$

is in K implies that $(q_2(t, a) - F_2(t, a))B_2 + (q_1(t, a) - F_1(t, a))B_1$ is also in K for all (t, a). Hence F', where $F'(t, a) = F_2(t, a)B_2 + F_1(t, a)B_1$, would be an extension of f, a contradiction of the fact that f has no extension which is C_2-dominated by q where C_2 is the induced wedge of type $C_2^{(4)}$ in the subspace V_2 spanned by B_1 and B_2.

If (W, K) is an ordered linear subspace of (V, C) then, since (V, C) is assumed to have the HBEP, a linear extension F of f will exist with range contained in V which is C-dominated by q. If W_3 is the subspace spanned by the union of the ranges of F and q with induced wedge $K_3 = W_3 \cap K$, it follows that (W_3, K_3) can be identified with (V_3, C_3) considered previously (with $C_3 = Q_3$) upon identifying B_1 with b_1, B_2 with b_2 and $B_{1.5}$ with $b_2 + b_3$ so that

$$K_3 = \{v \mid v = 0 \text{ or } v = a_1B_1 + a_{1.5}B_{1.5} + a_2B_2 \text{ where } a_2 > 0, \text{ or}$$
$$a_2 = 0, a_{1.5} > 0, \text{ or } a_2 = 0, a_{1.5} = 0, a_1 > 0\}.$$

The ordered linear subspace $(\tilde{W}, \tilde{K})$ of V spanned by W and $B_{1.5}$ has induced wedge $\tilde{K} = \tilde{W} \cap C$ of the same type as K. This will be proved when it is shown that if $\tilde{w} \in \tilde{W}$ and $\tilde{w} \neq 0$, then $\tilde{w} \in \tilde{K}$ if and only if $-\tilde{w} \notin \tilde{K}$. Let $\tilde{w} = \alpha B_2 + \beta B_{1.5} + \gamma B_1 + w''$ where $w'' \in W''$, the subspace bounding K', where K' is the wedge in W', the subspace bounding K. If $\alpha > 0$, then $\tilde{w} = u_1 + u_2$, where $u_1 = (\alpha/2)B_2 + \beta B_{1.5} \in K_3$ and $u_2 = (\alpha/2)B_2 + \gamma B_1 + w'' \in K$. Therefore $\tilde{w} \in K_3 + K \subset \tilde{K}$. If $\alpha < 0$, and $\tilde{w} \in K$, then from the previous sentence, $u = -\alpha B_2 - \beta B_{1.5}$

$-(\gamma+1)B_1-w''\in\tilde{K}$. Hence $u+\tilde{w}=-B_1\in\tilde{K}\cap W=K$, a contradiction, since $B_1\in K$.

If $\alpha=0$, $\beta>0$, then

$$\tilde{w} = (\beta B_{1.5} + (\gamma - 1)B_1) + (B_1 + w'') \in K_3 + K \subset \tilde{K}.$$

If $\alpha=0$, $\beta<0$ and $\tilde{w}\in\tilde{K}$, then from the previous sentence $u=-\beta B_{1.5}-(\gamma+1)B_1-w''\in\tilde{K}$. Therefore, $u+\tilde{w}=-B_1\in\tilde{K}\cap W=K$, again a contradiction.

If $\alpha=0$, $\beta=0$, then $\tilde{w}\in W$. Hence, if $\tilde{w}\neq 0$, $\tilde{w}\in K$ if and only if $-\tilde{w}\notin K$. Thus $\tilde{K}$ has the form asserted.

A Zorn's Lemma argument guarantees the existence of a maximal ordered linear subspace (W^*, K^*) of (V, C) whose induced wedge K^* is of the same form as K and which does not have the HBEP. The previous argument proves that $(W^*, K^*)=(V, C)$, a contradiction. Hence, no 2-dimensional cut of C by a subspace is of the type $C_2^{(4)}$. Thus, every 2-dimensional cut of C is closed and the Lemma and the Theorem are proved.

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