A NOTE ON REFLEXIVE BANACH SPACES

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The purpose of this note is to prove three well-known theorems concerning reflexive Banach spaces by using exact sequences.

Let $F$ denote either the field of real numbers or the field of complex numbers. Let $B$ be the category whose objects are Banach spaces over the field $F$ and whose morphisms are continuous linear maps $T: X \to Y$. As usual, $B(X, Y)$ denotes the set of all continuous linear maps from $X$ to $Y$. With the norm $|T| = \sup_{|x| \leq 1} |T(x)|$ for each $T$ in $B(X, Y)$, $B(X, Y)$ is a Banach space over $F$. Notice that if $T: X \to Y$ and $S: Y \to Z$ then $|ST| \leq |S||T|$. This implies that

$$B(T, Z): B(Y, Z) \to B(X, Z)$$

is a morphism in the category $B$ and $|B(T, Z)| \leq |T|$. As in [1], we let $B(X, F) = X^*$ and $B(T, F) = T^*$.

Let $Y$ be a closed subspace of the Banach space $X$. Let $i: Y \to X$ be the inclusion map. Then

$$0 \to Y \xrightarrow{i} X \xrightarrow{\rho} Z \to 0,$$

where $Z = X/Y$ and $\rho$ is the natural homomorphism, is an exact sequence in $B$. By the Hahn-Banach Theorem, the sequence

$$0 \leftarrow Y^* \leftarrow X^* \leftarrow Z^* \leftarrow 0$$

is exact. Therefore the sequence

$$0 \to Y^{**} \xrightarrow{i^{**}} X^{**} \xrightarrow{\rho^{**}} Z^{**} \to 0$$

is also exact. Clearly, we have the following commutative diagram

$$
\begin{array}{cccccccc}
& 0 & 0 & 0 \\
\downarrow & & & \downarrow \\
0 & \to & Y & \xrightarrow{i} & X & \xrightarrow{\rho} & Z & \to 0 \\
\downarrow n_1 & & \downarrow n_2 & & \downarrow n_3 \\
0 & \to & Y^{**} & \xrightarrow{i^{**}} & X^{**} & \xrightarrow{\rho^{**}} & Z^{**} & \to 0 \\
\end{array}
$$

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where \( n_1, n_2, n_3 \) are the natural embeddings and all the rows and columns are exact. Therefore, by "diagram chasing," we have the following commutative diagram (D):

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \\
\end{array} \quad \begin{array}{ccc}
Y & \rightarrow & X \\
i & \downarrow & \rho \\
Y** & \rightarrow & X** \\
\downarrow & \downarrow & \downarrow \\
Y**/Y & \rightarrow & X**/X \\
\downarrow & \downarrow & \downarrow \\
Z**/Z & \rightarrow & 0 \\
\end{array}
\]

where all the rows and columns are exact. In particular, the following sequence (E) is exact.

\[
(E) \quad 0 \rightarrow Y**/Y \rightarrow X**/X \rightarrow Z**/Z \rightarrow 0.
\]

We observe that the Banach space \( X \) is reflexive if and only if the Banach space \( X**/X \) in (D) (or (E)) is equal to 0.

**Theorem 1.** If \( X \) is a reflexive Banach space and \( Y \) is a closed subspace of \( X \), then \( Y \) is reflexive.

**Proof.** By the exactness of the sequence (E), we have \( X \) is reflexive \( \Rightarrow X**/X = 0 \Rightarrow Y**/Y = 0 \Rightarrow Y \) is reflexive.

**Theorem 2.** If \( X \) is a Banach space and \( Y \) is a closed subspace of \( X \), and if both \( Y \) and \( X/Y \) are reflexive, then \( X \) is reflexive.

**Proof.** Again by the exactness of the sequence (E), we have \( Y \) and \( X/Y \) are reflexive \( \Rightarrow Y**/Y = 0 \) and \( Z**/Z = 0 \Rightarrow X**/X = 0 \Rightarrow X \) is reflexive.

**Theorem 3.** If \( X \) is a reflexive Banach space and \( Y \) is a closed subspace of \( X \), then \( X/Y \) is reflexive.

**Proof.** By the exactness of the sequence (E), we have \( X \) is reflexive \( \Rightarrow X**/X = 0 \Rightarrow Z**/Z = 0 \Rightarrow Z = X/Y \) is reflexive.
ON TYPE I C*-ALGEBRAS

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1. Introduction. Recently, the author [4] proved the equivalence of type I C*-algebras and GCR C*-algebras without the assumption of separability. On the other hand, for separable type I C*-algebras, we have a simpler criterion as follows: a separable C*-algebra $A$ is of type I if and only if every irreducible image contains a nonzero compact operator.

It has been open whether or not this remains true when $A$ is not separable (cf. [1], [2], [3]).

In the present paper, we shall show that a C*-algebra $A$ is GCR if and only if every irreducible image contains a nonzero compact operator, so that by the author's previous theorem [4], the above problem is affirmative for arbitrary C*-algebra.

2. Theorem. In this section, we shall show the following theorem.

Theorem. A C*-algebra $A$ is of type I if and only if every irreducible image contains a nonzero compact operator.

Proof. Suppose that a C*-algebra $A$ is of type I, then it is GCR and so every irreducible image contains a nonzero compact operator (cf. [1], [2], [3], [4]).

Conversely suppose that every irreducible image of $A$ contains a nonzero compact operator. It is enough to assume that $A$ has the unit $I$. We shall assume that $A$ is not of type I. Then it is not GCR; then there is a separable nontype I C*-subalgebra $B$ of $A$ (cf. [2], [4]). Take a pure state $\phi$ on $B$ such that the image of $B$ under the irreducible *-representation $\{U_\phi, H_\phi\}$ of $B$ constructed via $\phi$ does not contain any nonzero compact operator, where $H_\phi$ is a Hilbert space.

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