1. Introduction. We shall use the symbol $H^\infty$ for the class of functions that are analytic and bounded in the unit disk $D$, and the symbol $A$ for the class of elements of $H^\infty$ that are continuous on the closure of $D$. For notational convenience, we shall often regard the unit circle $C$ and the interval $[0, 2\pi]$ as interchangeable. The purpose of this note is to prove the following theorem.

**Theorem 1.** Let the sequence $\{a_0, a_1, \ldots\}$ of complex numbers have the property that for each function $\sum b_n z^n$ in $H^\infty$ the limit

$$\lim_{r \to 1} \sum a_n b_n r^n$$

exists and is finite. Then there exists a function $\phi \in L^1(0, 2\pi)$ such that

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \phi(t)e^{int} dt = \hat{\phi}(n) \quad (n \geq 0).$$

The converse is also true.

This result was conjectured by A. E. Taylor in 1951 (see [7, p. 33]). The analogous proposition for two-sided sequences $\{\ldots, a_{-1}, a_0, a_1, \ldots\}$, with the space $H^\infty$ of bounded analytic functions replaced by $L^\infty$ and with Abel convergence replaced by Cesàro convergence, had been established by H. Steinhaus [6] in 1919.

To view the relations a little differently: if the sequence $\{a_n\}$ has the property described in our theorem, then it defines an additive, homogeneous functional on $H^\infty$. The theorem answers in the affirmative the question (raised in [5, p. 275]) whether this functional is continuous when the weak-star topology is imposed on $H^\infty$ as a subspace of $L^\infty$.

In §§2 and 3, we give a proof of Theorem 1. The crucial step is the construction of a Blaschke product whose absolute value is small on certain sets and is near to 1 (with closely controlled argument) on certain other sets. In §4, we examine Theorem 1 as a statement about multiplier transforms on certain sequence spaces.
2. **A Borel measure associated with \( \{a_n\} \).**

**Lemma 1.** If the sequence \( \{a_n\} \) satisfies the hypothesis in Theorem 1, then there exists a finite, complex-valued Borel measure \( \mu \) on \([0, 2\pi]\) such that

\[
a_n = \int e^{int} \, d\mu(t) = \bar{\mu}(n) \quad (n \geq 0).
\]

**Proof.** We may assume that the sequence \( \{a_n\} \) is bounded, since otherwise the limit (1) would be infinite for some absolutely convergent series \( \sum b_n \). For each \( r \) \((0 < r < 1) \) and each element \( f \) of \( A \), let

\[
\lambda_r(f) = \sum_{n=0}^{\infty} a_n b_n r^n.
\]

Since \( \{a_n\} \) is bounded, each \( \lambda_r \) is a bounded linear functional on \( A \), and by virtue of the Hahn-Banach theorem, we may extend it to a bounded linear functional (with the same norm) on the space of all complex-valued continuous functions on the unit circle \( C \). By a theorem of F. Riesz, there exists a finite Borel measure \( \mu_r \) such that

\[
\text{Var} \mu_r = \|\lambda_r\| \quad \text{and} \quad \lambda_r(f) = \int f \, d\mu_r \quad (f \in A).
\]

The hypothesis of Theorem 1 requires that for each \( f \) the set \( \{\lambda_r(f)\} \) \((0 < r < 1) \) is bounded. Hence, by the uniform-boundedness principle, the norms \( \|\lambda_r\| \) are bounded. By the weak-star compactness of measures, there exists a measure \( \mu \) such that

\[
\lim \int f \, d\mu_r = \int f \, d\mu \quad (f \in A).
\]

To complete the proof of Lemma 1, we take successively the functions \( f(z) = z^n \) \((n = 0, 1, \ldots)\).

3. **Absolute continuity of \( \mu \).** The absolute continuity of the measure \( \mu \) constructed in §2 is a consequence of the following theorem.

**Theorem 2.** Let \( \mu \) be a finite, complex-valued Borel measure on \( |z| = 1 \) such that

\[
\lim_{r \to 1} \int f(re^{it}) \, d\mu(t)
\]

exists for all Blaschke products \( f \). Then \( \mu \) is absolutely continuous.
Remark 1. Theorem 2 is a generalization of the following theorem of F. and M. Riesz: A measure whose Fourier-Stieltjes coefficients vanish on one side is absolutely continuous. Indeed, if \( \hat{\mu}(n) = 0 \) \( (n > 0) \), then \( \int fd\mu = 0 \) for all \( f \in H^\infty \) (in particular, for all Blaschke products), and thus the limit (2) exists.

Remark 2. Let \( S_\mu \) denote the collection of all \( f \in H^\infty \) for which the limit (2) exists. Clearly, \( S_\mu \) is a vector subspace of \( H^\infty \). Further, it is a closed set in the metric of \( H^\infty \) (the proof of this leads to a double limit, but one of the limits is uniform over the whole open disk, not merely over compact subsets). By the hypothesis of Theorem 2, \( S_\mu \) contains all Blaschke products. The following question (see [4, p. 855, Problem C]) remains open. Does there exist a proper closed vector subspace of \( H^\infty \) that contains all Blaschke products?

Remark 3. We do not know whether Theorem 1 remains valid if in the hypothesis we merely require that the limit (1) exists for all Blaschke products \( f = \sum b_n z^n \), rather than for all \( f \in H^\infty \).

Proof of Theorem 2. We must show that \( \mu(F) = 0 \) for each Borel set \( F \) of Lebesgue measure 0 (on \( C \) or on \( [0, 2\pi] \)). By the regularity of the measure, it is enough to prove the proposition for closed sets \( F \). Our proof will proceed by contradiction: we shall assume that \( \mu(F) \neq 0 \) for some (fixed) closed set \( F \) of Lebesgue measure 0.

We shall require several lemmas. In each lemma, we assume the hypothesis of Theorem 2. For \( 0 < r < 1 \), we use the notation \( f_r(z) = f(rz) \).

Lemma 2. Without loss of generality, we may assume that
\[
\Re \mu(E) \geq 0 \quad \text{and} \quad \Im \mu(E) \geq 0
\]
for all Borel sets \( E \subseteq F \).

Proof. This follows from the Jordan decomposition theorem for measures: the measure \( \Re \mu \), restricted to the set \( F \), is the difference of two positive measures that live on disjoint subsets of \( F \); a corresponding statement applies to \( \Im \mu \). Thus, if \( F \) does not have the property in the lemma, we simply replace it with a subset \( E \) on which each of \( \Re \mu(E) \) and \( \Im \mu(E) \) has constant sign. Multiplication of \( \mu(E) \) with the appropriate power of \( i \) then gives the desired result.

Lemma 3. The limit
\[
\lim_{r \to 1} \int_{F_p} f_r d\mu
\]
exists for every Blaschke product \( f \).
Proof. If \( f \) is a Blaschke product, then

\[
\lim_{r \to 1} \int f_r h d\mu
\]

exists for each \( h \) in the space \( A \) of uniformly continuous analytic functions. This follows immediately from the hypothesis if \( h \) is a power of \( z \); the general case then follows from Remark 2, since the polynomials are dense in \( A \).

Next we observe that \( \lim \int f_r h d\mu \) exists for each \( h \in A \), since

\[
\int f_r \cdot (h_r - h) d\mu \leq ||f|| \cdot ||h_r - h|| \text{ Var}(\mu) \to 0.
\]

Next we choose a function \( g \in A \) such that \( g = 1 \) on \( F \) and \( |g| < 1 \) on the complement \( F' \) of \( F \) (concerning the existence of such a function, see [2, Chapter VI, p. 81]). Then

\[
\int f_r d\mu = \int f_r g^n d\mu - \int f_r g^n d\mu.
\]

For each \( n \), the first term on the right has a limit, as we have just seen; for large \( n \), we can make the second term on the right arbitrarily small (uniformly in \( r \)), since

\[
\left| \int f_r g^n d\mu \right| \leq ||f|| \int |g|^n d\mu
\]

and the right member tends to zero as \( n \to \infty \), by bounded convergence.

Lemma 4. There exist closed sets \( I_p \) (\( p = 1, 2, \ldots \)) such that

(i) each set \( I_p \) is the union of a finite number \( m_p \) of closed arcs \( I_{p j} \) (\( j = 1, 2, \ldots, m_p \)), all having the same length \( d_p \),

(ii) \( m_p d_p < 1/p^2 \),

(iii) \( F = \bigcap I_p \).

Proof. Let \( F_n \) denote the set of points whose distance from \( F \) is at most \( 2\pi/n \). Then

\[
F_1 \supset F_2 \supset \cdots \supset F_n \supset F,
\]

and therefore \( |F_n| \to 0 \) (by \( |M| \) we denote the Lebesgue measure of the set \( M \)).

For each natural number \( n \), we divide the unit circle into \( 2n \) equal closed subarcs, and we denote by \( J_n \) the union of the arcs that meet...
the set $F$. Then $J_n \subset F_n$, and therefore $|J_n| \to 0$. Finally, we extract a subsequence $\{J_{n_p}\}$ such that $|J_{n_p}| < 1/p^2$, and we write $I_p = J_{n_p}$.

**Lemma 5.** Let $r < 1$ and $\epsilon > 0$ be fixed. Then there exist a number $s (r < s < 1)$ and a finite Blaschke product $b(z)$ whose $m$ zeros all lie on the circle $|z| = s$, such that

(i) $(1 - s)m < \epsilon$,
(ii) $|b(se^{it})| < \epsilon$ ($e^{it} \in F$),
(iii) $|b(z) - 1| < \epsilon (|z| \leq r)$.

**Proof.** We define the positive number $s_p$ by the equation

$$1 - s_p^2 = pd_p.$$ 

Then $0 < s_p < 1$ and $s_p \to 1$. Let $z_{pj}$ denote the midpoint of the arc $s_p I_{pj}$. For each $z$ on this arc, $|z_{pj} - z| < dp/2$, and therefore

$$|z_{pj} - z|/(1 - z_{pj}^2) \leq dp/2(1 - s_p^2) = 1/2p.$$ 

Hence the finite product

$$b_p(z) = \prod_{1 \leq j \leq m_p} \frac{|z_{pj}|}{z_{pj}} \frac{z_{pj} - z}{1 - z_{pj}^2}$$

satisfies the inequality $|b_p(z)| < 1/2p$ for all $z \in s_p I_p$.

The number of factors of $b_p$ is $m_p$, and

$$(3) \quad m_p(1 - s_p) < m_p(1 - s_p^2) = pm_pdp < 1/p.$$ 

Thus the finite Blaschke product $b_p(z)$ satisfies (i) and (ii) for all $p > 1/\epsilon$.

To discuss (iii), we need the formula

$$|a| \cdot \frac{a - z}{1 - az} = 1 - \epsilon_a(z),$$

where

$$\epsilon_a(z) = (a + |a| z)(1 - |a|)/a(1 - \bar{a}z).$$

We shall assume that $1/2 < |a| < 1$. Then

$$|\epsilon_a(z)| \leq 4(1 - |a|)/|1 - \bar{a}z|,$$

and for $|z| \leq r$,

$$|\epsilon_a(z)| \leq 4(1 - |a|)/(1 - r|a|) \leq 4(1 - |a|)/(1 - r).$$

Now consider the finite product $b_p(z)$, with $p$ large enough so that $s_p > 1/2$ and $s_p > r$. To simplify the notation, we write


\[ b_p(z) = \prod (1 - e_\alpha(z)), \]

where the product has \( m_p \) factors.

For \(|z| \leq r\), we have the inequalities

\[ |b_p(z) - 1| \leq \prod (1 + |e_\alpha(z)|) - 1 \leq (\exp \sum |e_\alpha(z)|) - 1. \]

By (6),

\[ \sum |e_\alpha(z)| \leq 4m_p(1 - s_p)/(1 - r), \]

and therefore (3) implies that \( b_p \) satisfies (iii) if \( p \) is large enough.

**Lemma 6.** Suppose \( b_0(z) \) is a finite Blaschke product, \( \epsilon > 0 \), and \( 0 < r < 1 \). Then we can find a number \( s \) \((r < s < 1)\) and a finite Blaschke product \( b(z) \) whose \( m \) zeros all lie on the circle \(|z| = s\), such that

(i) \( (1 - s)m < \epsilon \),

(ii) \( \left| \int_{F_0} b_0(se^{it})b(se^{it})d\mu(t) \right| > |\mu(F)|/24, \)

(iii) \( |b(z) - 1| < \epsilon \left( |z| \leq r \right) \).

**Proof.** We divide the unit circle into eight equal open arcs \( J_1, \cdots, J_8 \) whose end points all lie in the complement of \( b_0(F) \) (this is possible, since \( b_0(F) \) is nowhere dense). We then define the sets

\[ E_k = \{ e^{it}; b_0(e^{it}) \in J_k \} \quad (k = 1, 2, \cdots, 8). \]

These sets are open and disjoint, and they cover \( F \). We write \( F_k = F \cap E_k \). Without loss of generality, we may assume that \( |\mu(F_1)| \geq |\mu(F)|/8. \)

We shall now use the geometrically obvious fact that if \( w_m = r_m e_m^u \) \((m = 1, 2, \cdots, n; 0 \leq t_m \leq 3\pi/4)\), then \( |\sum w_m| \geq c \sum |w_m| \), where

\[ c = \frac{1}{2} \left| 2 - \sqrt{2} \right|^{1/2} > 1/3. \]

Together with Lemma 2, it implies that

\[ \left| \int_{F_1} b_0(e^{it})d\mu(t) \right| > |\mu| (F_1)/3 \geq |\mu(F_1)|/3 \geq |\mu(F)|/24, \]

where \( |\mu| (F_1) \) denotes the total variation of \( \mu \) over \( F_1 \).

Let \( s_p \) be defined as in the proof of Lemma 5. If \( s_p \) is near enough to \( 1 \), then

\[ \left| \int_{F_1} b_0(s_p e^{it})d\mu(t) \right| > |\mu(F)|/24, \]

by the uniform continuity of \( b_0(z) \).

Let \( \delta = \text{dist} (F_1, F \setminus F_1) \). With the notation of Lemma 4, choose \( \rho \)
large enough so that \( d_p < \delta \). In (4), let \( |a| = s_p \) and \( z = au \), with \( |u| = 1 \). Since \( |1 - u| \leq 2 \left| 1 - s_p^2 u \right| \), it follows from (5) that

\[
|\varepsilon_a(z)| \leq 4(1 - s_p)/\left| 1 - s_p^2 u \right| \leq 8(1 - s_p)/\left| 1 - u \right|.
\]

For a fixed \( p \), we select among the arcs \( I_{pj} \) (\( j = 1, \ldots, m_p \)) those that do not meet \( F_i \) (since \( d_p < \delta \), none of the arcs \( I_{pj} \) meeting \( F_i \) meets any of the other sets \( F_k \)).

We denote the midpoints of the selected arcs by \( z_{pj} \), and we form the finite Blaschke product

\[
b_p(z) = \prod \frac{|z_{pj}|}{z_{pj}} \frac{z_{pj} - z}{1 - \bar{z}_{pj} z}.
\]

Since the product has at most \( m_p \) factors, it satisfies condition (i) of the lemma. Just as in the proof of Lemma 5, we have the inequality

\[
|b_p(z)| < 1/2p \quad \text{for} \quad z/s_p \in (F/F_i).
\]

If \( z/s_p \in F_i \) and \( a \) denotes the midpoint \( z_{pj} \) of some selected interval, we can write \( z = au \), where \( |u| = 1 \) and \( |\arg u| > \delta/2 \). Therefore (9) implies that \( |\varepsilon_a(z)| \leq c(1 - s_p)/\delta \), for some constant \( c \), and therefore

\[
\sum |\varepsilon_a(z)| \leq cm_p(1 - s_p)/\delta.
\]

By (7) and (11) we see that if \( p \) is large enough, then \( b_p(z) \) is arbitrarily near to 1 on \( s_pF_i \) and arbitrarily small on \( s_p(F/F_i) \); therefore, in view of (8), condition (ii) of the lemma is satisfied.

Finally, just as in the proof of Lemma 5, we can show that \( b_p(z) \) satisfies condition (iii) if \( p \) is large enough. This completes the proof of Lemma 6.

To prove Theorem 2, we use alternately Lemmas 5 and 6 to construct an infinite Blaschke product

\[
f(z) = \prod b_n(z)
\]

with the following properties:

(i) all the zeros of \( b_n(z) \) lie on a circle \( |z| = r_n \);
(ii) if \( n \) is odd and \( e^{it} \in F \), then \( |b_n(r_n e^{it})| < 1/n \);
(iii) if \( n \) is even, then

\[
\left| \int_{F} f_n(r_n e^{it})d\mu(t) \right| > |\mu(F)|/24,
\]

where \( f_n = b_1 \cdots b_n \);
(iv) for large values of \( n \), the product \( b_{n+1}b_{n+2} \cdots \) is close to 1 on the disk \( |z| \leq r_n \).
Clearly, the function \( f \) satisfies the inequalities

\[
\left| \int f(r_n e^{it}) d\mu(t) \right| \leq \frac{|\mu(F)|}{n} \quad (n \text{ odd}),
\]
\[
\left| \int f(r_n e^{it}) d\mu(t) \right| > \frac{|\mu(F)|}{24} \quad (n \text{ even}).
\]

Since this contradicts Lemma 3, the proof of Theorem 2 is complete.

The converse part of Theorem 1 is well known, and we merely indicate a proof. If \( \phi \in L^1 \) and \( f \in L^\infty \) (in particular, if \( f \in H^\infty \)), then the convolution \( \phi \ast f \) is a continuous function. The Abel mean of the Fourier series of a continuous function converges to the function uniformly; in particular, it converges at the point \( z = 1 \). This is precisely the assertion that the limit (1) exists.

4. Multiplier transforms. Let \( X \) and \( Y \) be two spaces of sequences, and let \( \{\lambda_n\} \) be a fixed sequence.

**Definition.** \( \{\lambda_n\} \) is of class \((X, Y)\) if \( \{\lambda_n a_n\} \in Y \) for each \( \{a_n\} \in X \).

Let \( R \) denote the space of bounded analytic functions in \( D \) whose radial limit exists on every radius. We may regard \( R \) and \( A \) (see §1) as spaces of sequences (Taylor coefficients).

Let \( L^+ \) denote the space of sequences that constitute one side of the sequence of Fourier coefficients of some integrable function, and let \( S^+ \) denote the space of sequences that constitute one side of the sequence of Fourier-Stieltjes coefficients of some measure. That is, let \( \{a_n\} \in L^+ \) if \( \{a_n\} \in S^+ \) if and only if there exists a function \( \phi \in L \) (a measure \( \mu \)) such that

\[
a_n = \hat{\phi}(n) \quad (a_n = \hat{\mu}(n)) \quad \text{for } n \geq 0.
\]

From Theorem 1 we obtain the following result.

**Theorem 3.** \((H^\infty, R) = (H^\infty, A) = (S^+, L^+) = L^+\).

**Proof.** The equation \((H^\infty, R) = L^+\) is merely a restatement of Theorem 1. Indeed, the existence of the limit (1) is precisely the existence of the radial limit along the unit interval; the existence of the limit along any other radius follows by rotation.

Clearly, \((H^\infty, A) \subset (H^\infty, R) = L^+.\) Conversely, the convolution of a function in \( L^1 \) with a function in \( L^\infty \) is continuous.

The identity sequence \( \{1, 1, \cdots\} \) is in \( S^+ \), and therefore \((S^+, L^+) \subset L^+.\) The reverse inclusion follows from the fact that \((S, L) = L\) for two-sided sequences (see Zygmund [8, Chapter IV, Theorem 11.10]). This completes the proof of the theorem.
The third equation in Theorem 3, \((S_+, L_+) = L_+\), is in some sense a dual of the second equation, \((H^\infty, A) = L_+\). As we have just seen, it is easy to prove. However, we do not know how to establish the second equation without going through Theorem 1. Perhaps this is to be expected; for although \(S_+\) is the conjugate space of the Banach space \(A\), the space \(L_+\) only becomes the full dual of \(H^\infty\) when \(H^\infty\) is given the weak-star topology; as we mentioned in the Introduction, Theorem 1 may be regarded as a statement about weak-star continuous linear functionals on \(H^\infty\).

In conclusion, we mention a conjecture that seems to be difficult to settle (it implies Theorem 1).

**Conjecture.** Let \(\{\phi_n\}\) be a sequence of elements in \(L\) such that
\[
\lim_{n \to \infty} \int \phi_n f \text{ exists for each } f \in H^\infty.
\]
Then there exists a \(\phi \in L\) such that
\[
\lim_{n \to \infty} \int \phi_n f = \int \phi f \quad \text{for all } f \in H^\infty.
\] (13)

To state the problem differently: is the quotient Banach space \(L/H\) weakly sequentially complete? (This question was raised in [1, pp. 180–181].) The analogous result for \(L\) instead of \(L/H\) (that is, with \(L^\infty\) instead of \(H^\infty\) in the statement above) was proved by Steinhaus [6] (see also the proof in Zygmund [8, Chapter IV, Theorem 9.13]).

In the paper immediately following this one, Kahane [3] gives a partial affirmative answer to the conjecture. He shows that there exists a \(\phi \in L\) such that the relation (13) holds for all \(f \in A\). This is enough to imply Theorem 1.

**References**