# BOUNDED ANALYTIC FUNCTIONS AND ABSOLUTELY CONTINUOUS MEASURES[1]

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1. **Introduction.** We shall use the symbol $H^\infty$ for the class of functions that are analytic and bounded in the unit disk $D$, and the symbol $A$ for the class of elements of $H^\infty$ that are continuous on the closure of $D$. For notational convenience, we shall often regard the unit circle $C$ and the interval $[0, 2\pi]$ as interchangeable. The purpose of this note is to prove the following theorem.

THEOREM 1. *Let the sequence $\{a_0, a_1, \cdots\}$ of complex numbers have the property that for each function $\sum b_n z^n$ in $H^\infty$ the limit*

$$\lim_{r\to 1} \sum a_n b_n r^n \tag{1}$$

*exists and is finite. Then there exists a function $\phi \in L^1(0, 2\pi)$ such that*

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \phi(t)e^{int}\,dt = \hat{\phi}(n) \qquad (n \geqq 0).$$

*The converse is also true.*

This result was conjectured by A. E. Taylor in 1951 (see [7, p. 33]). The analogous proposition for two-sided sequences $\{\cdots, a_{-1}, a_0, a_1, \cdots\}$, with the space $H^\infty$ of bounded analytic functions replaced by $L^\infty$ and with Abel convergence replaced by Cesàro convergence, had been established by H. Steinhaus [6] in 1919.

To view the relations a little differently: if the sequence $\{a_n\}$ has the property described in our theorem, then it defines an additive, homogeneous functional on $H^\infty$. The theorem answers in the affirmative the question (raised in [5, p. 275]) whether this functional is continuous when the weak-star topology is imposed on $H^\infty$ as a subspace of $L^\infty$.

In §§2 and 3, we give a proof of Theorem 1. The crucial step is the construction of a Blaschke product whose absolute value is small on certain sets and is near to 1 (with closely controlled argument) on certain other sets. In §4, we examine Theorem 1 as a statement about multiplier transforms on certain sequence spaces.

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## 2. A Borel measure associated with $\{a_n\}$.

LEMMA 1. *If the sequence* $\{a_n\}$ *satisfies the hypothesis in Theorem* 1, *then there exists a finite, complex-valued Borel measure* $\mu$ *on* $[0, 2\pi]$ *such that*

$$a_n = \int e^{int}\, d\mu(t) = \hat{\mu}(n) \qquad (n \geqq 0).$$

PROOF. We may assume that the sequence $\{a_n\}$ is bounded, since otherwise the limit (1) would be infinite for some absolutely convergent series $\sum b_n$. For each $r$ $(0<r<1)$ and each element $f(z)=\sum b_n z^n$ of $A$, let

$$\lambda_r(f) = \sum_0^\infty a_n b_n r^n.$$

Since $\{a_n\}$ is bounded, each $\lambda_r$ is a bounded linear functional on $A$, and by virtue of the Hahn-Banach theorem, we may extend it to a bounded linear functional (with the same norm) on the space of all complex-valued continuous functions on the unit circle $C$. By a theorem of F. Riesz, there exists a finite Borel measure $\mu_r$ such that

$$\operatorname{Var} \mu_r = \|\lambda_r\| \quad \text{and} \quad \lambda_r(f) = \int f d\mu_r \qquad (f \in A).$$

The hypothesis of Theorem 1 requires that for each $f$ the set $\{\lambda_r(f)\}$ $(0<r<1)$ is bounded. Hence, by the uniform-boundedness principle, the norms $\|\lambda_r\|$ are bounded. By the weak-star compactness of measures, there exists a measure $\mu$ such that

$$\lim \int f d\mu_r = \int f d\mu \qquad (f \in A).$$

To complete the proof of Lemma 1, we take successively the functions $f(z)=z^n$ $(n=0, 1, \cdots)$.

## 3. Absolute continuity of $\mu$.

The absolute continuity of the measure $\mu$ constructed in §2 is a consequence of the following theorem.

THEOREM 2. *Let* $\mu$ *be a finite, complex-valued Borel measure on* $|z|=1$ *such that*

$$\lim_{r\to 1} \int f(re^{it})\, d\mu(t) \tag{2}$$

*exists for all Blaschke products* $f$. *Then* $\mu$ *is absolutely continuous.*

REMARK 1. Theorem 2 is a generalization of the following theorem of F. and M. Riesz: *A measure whose Fourier-Stieltjes coefficients vanish on one side is absolutely continuous.* Indeed, if $\hat{\mu}(n)=0$ $(n>0)$, then $\int f d\mu=0$ for all $f\in H^\infty$ (in particular, for all Blaschke products), and thus the limit (2) exists.

REMARK 2. Let $S_\mu$ denote the collection of all $f\in H^\infty$ for which the limit (2) exists. Clearly, $S_\mu$ is a vector subspace of $H^\infty$. Further, it is a closed set in the metric of $H^\infty$ (the proof of this leads to a double limit, but one of the limits is uniform over the whole open disk, not merely over compact subsets). By the hypothesis of Theorem 2, $S_\mu$ contains all Blaschke products. The following question (see [**4**, p. 855, Problem C]) remains open. Does there exist a proper closed vector subspace of $H^\infty$ that contains all Blaschke products?

REMARK 3. We do not know whether Theorem 1 remains valid if in the hypothesis we merely require that the limit (1) exists for all Blaschke products $f=\sum b_n z^n$, rather than for all $f\in H^\infty$.

PROOF OF THEOREM 2. We must show that $\mu(F)=0$ for each Borel set $F$ of Lebesgue measure 0 (on $C$ or on $[0, 2\pi]$). By the regularity of the measure, it is enough to prove the proposition for closed sets $F$. Our proof will proceed by contradiction: we shall assume that $\mu(F)\neq 0$ for some (fixed) closed set $F$ of Lebesgue measure 0.

We shall require several lemmas. In each lemma, we assume the hypothesis of Theorem 2. For $0<r<1$, we use the notation $f_r(z)=f(rz)$.

LEMMA 2. *Without loss of generality, we may assume that*

$$\Re\mu(E)\geqq 0 \quad \textit{and} \quad \Im\mu(E)\geqq 0$$

*for all Borel sets* $E\subset F$.

PROOF. This follows from the Jordan decomposition theorem for measures: the measure $\Re\mu$, restricted to the set $F$, is the difference of two positive measures that live on disjoint subsets of $F$; a corresponding statement applies to $\Im\mu$. Thus, if $F$ does not have the property in the lemma, we simply replace it with a subset $E$ on which each of $\Re\mu(E)$ and $\Im\mu(E)$ has constant sign. Multiplication of $\mu(E)$ with the appropriate power of $i$ then gives the desired result.

LEMMA 3. *The limit*

$$\lim_{r\to 1}\int_F f_r d\mu$$

*exists for every Blaschke product* $f$.

PROOF. If $f$ is a Blaschke product, then

$$\lim_{r\to 1}\int f_r h_r d\mu$$

exists for each $h$ in the space $A$ of uniformly continuous analytic functions. This follows immediately from the hypothesis if $h$ is a power of $z$; the general case then follows from Remark 2, since the polynomials are dense in $A$.

Next we observe that $\lim \int f_r\, h d\mu$ exists for each $h\in A$, since

$$\left|\int f_r\cdot(h_r - h)d\mu\right| \leqq \|f\|\,\|h_r - h\|\ \mathrm{Var}(\mu)\to 0.$$

Next we choose a function $g\in A$ such that $g=1$ on $F$ and $|g|<1$ on the complement $F'$ of $F$ (concerning the existence of such a function, see [2, Chapter VI, p. 81]). Then

$$\int_F f_r d\mu = \int f_r g^n d\mu - \int_{F'} f_r g^n d\mu.$$

For each $n$, the first term on the right has a limit, as we have just seen; for large $n$, we can make the second term on the right arbitrarily small (uniformly in $r$), since

$$\left|\int_{F'} f_r g^n d\mu\right| \leqq \|f\| \int_{F'} |g|^n d|\mu|$$

and the right member tends to zero as $n\to\infty$, by bounded convergence.

LEMMA 4. *There exist closed sets* $I_p$ ($p=1, 2, \cdots$) *such that*

(i) *each set* $I_p$ *is the union of a finite number* $m_p$ *of closed arcs* $I_{pj}$ ($j=1, 2, \cdots, m_p$), *all having the same length* $d_p$,

(ii) $m_p d_p < 1/p^2$,

(iii) $F=\bigcap I_p$.

PROOF. Let $F_n$ denote the set of points whose distance from $F$ is at most $2\pi/n$. Then

$$F_1\supset F_2\supset \cdots \supset F, \qquad \bigcap F_n = F,$$

and therefore $|F_n|\to 0$ (by $|M|$ we denote the Lebesgue measure of the set $M$).

For each natural number $n$, we divide the unit circle into $2n$ equal closed subarcs, and we denote by $J_n$ the union of the arcs that meet the set $F$. Then $J_n \subset F_n$, and therefore $|J_n| \to 0$. Finally, we extract a subsequence $\{J_{n_p}\}$ such that $|J_{n_p}| < 1/p^2$, and we write $I_p = J_{n_p}$.

LEMMA 5. *Let $r<1$ and $\epsilon>0$ be fixed. Then there exist a number $s$ $(r<s<1)$ and a finite Blaschke product $b(z)$ whose $m$ zeros all lie on the circle $|z|=s$, such that*

(i) $(1-s)m < \epsilon$,
(ii) $|b(se^{it})| < \epsilon$ $(e^{it} \in F)$,
(iii) $|b(z)-1| < \epsilon$ $(|z| \leqq r)$.

PROOF. We define the positive number $s_p$ by the equation

$$1 - s_p^2 = pd_p.$$

Then $0 < s_p < 1$ and $s_p \to 1$. Let $z_{pj}$ denote the midpoint of the arc $s_p I_{pj}$. For each $z$ on this arc, $|z_{pj} - z| < d_p/2$, and therefore

$$|(z_{pj} - z)/(1 - \bar{z}_{pj} z)| \leqq d_p/2(1 - s_p^2) = 1/2p.$$

Hence the finite product

$$b_p(z) = \prod_{1 \leqq j \leqq m_p} \frac{|z_{pj}|}{z_{pj}} \frac{z_{pj} - z}{1 - \bar{z}_{pj} z}$$

satisfies the inequality $|b_p(z)| < 1/2p$ for all $z \in s_p I_p$.

The number of factors of $b_p$ is $m_p$, and

$$m_p(1 - s_p) < m_p(1 - s_p^2) = pm_p d_p < 1/p. \tag{3}$$

Thus the finite Blaschke product $b_p(z)$ satisfies (i) and (ii) for all $p > 1/\epsilon$.

To discuss (iii), we need the formula

$$\frac{|a|}{a} \cdot \frac{a - z}{1 - \bar{a}z} = 1 - \epsilon_a(z), \tag{4}$$

where

$$\epsilon_a(z) = (a + |a| z)(1 - |a|)/a(1 - \bar{a}z).$$

We shall assume that $1/2 < |a| < 1$. Then

$$|\epsilon_a(z)| \leqq 4(1 - |a|)/|1 - \bar{a}z|, \tag{5}$$

and for $|z| \leqq r$,

$$|\epsilon_a(z)| \leqq 4(1 - |a|)/(1 - r|a|) \leqq 4(1 - |a|)/(1 - r). \tag{6}$$

Now consider the finite product $b_p(z)$, with $p$ large enough so that $s_p > 1/2$ and $s_p > r$. To simplify the notation, we write

$$b_p(z) = \prod (1 - \epsilon_a(z)),$$

where the product has $m_p$ factors.

For $|z| \leqq r$, we have the inequalities

$$(7) \quad | b_p(z) - 1 | \leqq \prod (1 + | \epsilon_a(z) | ) - 1 \leqq (\exp \sum | \epsilon_a(z) | ) - 1.$$

By (6),

$$\sum | \epsilon_a(z) | \leqq 4m_p(1 - s_p)/(1 - r),$$

and therefore (3) implies that $b_p$ satisfies (iii) if $p$ is large enough.

LEMMA 6. *Suppose $b_0(z)$ is a finite Blaschke product, $\epsilon > 0$, and $0 < r < 1$. Then we can find a number $s$ $(r < s < 1)$ and a finite Blaschke product $b(z)$ whose $m$ zeros all lie on the circle $|z| = s$, such that*

(i) $(1 - s)m < \epsilon$,

(ii) $| \int_F b_0(se^{it})b(se^{it})d\mu(t) | > |\mu(F)|/24$,

(iii) $|b(z) - 1| < \epsilon$ $(|z| \leqq r)$.

PROOF. We divide the unit circle into eight equal open arcs $J_1, \cdots, J_8$ whose end points all lie in the complement of $b_0(F)$ (this is possible, since $b_0(F)$ is nowhere dense). We then define the sets

$$E_k = \{ e^{it} : b_0(e^{it}) \in J_k \} \qquad (k = 1, 2, \cdots, 8).$$

These sets are open and disjoint, and they cover $F$. We write $F_k = F \cap E_k$. Without loss of generality, we may assume that $|\mu(F_1)| \geqq |\mu(F)|/8$.

We shall now use the geometrically obvious fact that if $w_m = r_m e_m^{it}$ $(m = 1, 2, \cdots, n; 0 \leqq t_m \leqq 3\pi/4)$, then $| \sum w_m | \geqq c \sum |w_m|$, where

$$c = \tfrac{1}{2}[2 - \sqrt{2}]^{1/2} > 1/3.$$

Together with Lemma 2, it implies that

$$\left| \int_{F_1} b_0(e^{it}) d\mu(t) \right| > |\mu|(F_1)/3 \geqq |\mu(F_1)|/3 \geqq |\mu(F)|/24,$$

where $|\mu|(F_1)$ denotes the total variation of $\mu$ over $F_1$.

Let $s_p$ be defined as in the proof of Lemma 5. If $s_p$ is near enough to 1, then

$$(8) \qquad \left| \int_{F_1} b_0(s_p e^{it}) d\mu(t) \right| > |\mu(F)|/24,$$

by the uniform continuity of $b_0(z)$.

Let $\delta = \operatorname{dist}(F_1, F \backslash F_1)$. With the notation of Lemma 4, choose $p$

large enough so that $d_p < \delta$. In (4), let $|a| = s_p$ and $z = au$, with $|u| = 1$. Since $|1-u| \leqq 2|1-s_p^2 u|$, it follows from (5) that

$$(9) \qquad |\epsilon_a(z)| \leqq 4(1-s_p)/|1-s_p^2 u| \leqq 8(1-s_p)/|1-u|.$$

For a fixed $p$, we select among the arcs $I_{pj}$ ($j=1, \cdots, m_p$) those that do not meet $F_1$ (since $d_p < \delta$, none of the arcs $I_{pj}$ meeting $F_1$ meets any of the other sets $F_k$).

We denote the midpoints of the selected arcs by $z_{pj}$, and we form the finite Blaschke product

$$(10) \qquad b_p(z) = \prod \frac{|z_{pj}|}{z_{pj}} \frac{z_{pj} - z}{1 - \bar{z}_{pj} z}.$$

Since the product has at most $m_p$ factors, it satisfies condition (i) of the lemma. Just as in the proof of Lemma 5, we have the inequality

$$(11) \qquad |b_p(z)| < 1/2p \qquad \text{for } z/s_p \in (F \backslash F_1).$$

If $z/s_p \in F_1$ and $a$ denotes the midpoint $z_{pj}$ of some selected interval, we can write $z = au$, where $|u| = 1$ and $|\arg u| > \delta/2$. Therefore (9) implies that $|\epsilon_a(z)| \leqq c(1-s_p)/\delta$, for some constant $c$, and therefore

$$\sum |\epsilon_a(z)| \leqq c m_p (1-s_p)/\delta.$$

By (7) and (11) we see that if $p$ is large enough, then $b_p(z)$ is arbitrarily near to 1 on $s_p F_1$ and arbitrarily small on $s_p(F \backslash F_1)$; therefore, in view of (8), condition (ii) of the lemma is satisfied.

Finally, just as in the proof of Lemma 5, we can show that $b_p(z)$ satisfies condition (iii) if $p$ is large enough. This completes the proof of Lemma 6.

To prove Theorem 2, we use alternately Lemmas 5 and 6 to construct an infinite Blaschke product

$$f(z) = \prod b_n(z)$$

with the following properties:

(i) all the zeros of $b_n(z)$ lie on a circle $|z| = r_n$;

(ii) if $n$ is odd and $e^{it} \in F$, then $|b_n(r_n e^{it})| < 1/n$;

(iii) if $n$ is even, then

$$\left| \int_F f_n(r_n e^{it}) d\mu(t) \right| > |\mu(F)|/24,$$

where $f_n = b_1 \cdots b_n$;

(iv) for large values of $n$, the product $b_{n+1} b_{n+2} \cdots$ is close to 1 on the disk $|z| \leqq r_n$.

Clearly, the function $f$ satisfies the inequalities

$$\left| \int_F f(r_n e^{it}) d\mu(t) \right| \leqq |\mu(F)|/n \qquad (n \text{ odd}),$$

$$\left| \int_F f(r_n e^{it}) d\mu(t) \right| > |\mu(F)|/24 \qquad (n \text{ even}).$$

Since this contradicts Lemma 3, the proof of Theorem 2 is complete.

The converse part of Theorem 1 is well known, and we merely indicate a proof. If $\phi \in L^1$ and $f \in L^\infty$ (in particular, if $f \in H^\infty$), then the convolution $\phi * f$ is a continuous function. The Abel mean of the Fourier series of a continuous function converges to the function uniformly; in particular, it converges at the point $z=1$. This is precisely the assertion that the limit (1) exists.

4. **Multiplier transforms.** Let $X$ and $Y$ be two spaces of sequences, and let $\{\lambda_n\}$ be a fixed sequence.

Definition. $\{\lambda_n\}$ *is of class* $(X, Y)$ if $\{\lambda_n a_n\} \in Y$ for each $\{a_n\} \in X$.

Let $R$ denote the space of bounded analytic functions in $D$ whose radial limit exists on every radius. We may regard $R$ and $A$ (see §1) as spaces of sequences (Taylor coefficients).

Let $L_+$ denote the space of sequences that constitute one side of the sequence of Fourier coefficients of some integrable function, and let $S_+$ denote the space of sequences that constitute one side of the sequence of Fourier-Stieltjes coefficients of some measure. That is, let $\{a_n\} \in L_+$ ($\{a_n\} \in S_+$) if and only if there exists a function $\phi \in L$ (a measure $\mu$) such that

$$a_n = \hat{\phi}(n) \quad (a_n = \hat{\mu}(n)) \qquad \text{for } n \geqq 0.$$

From Theorem 1 we obtain the following result.

Theorem 3. $(H^\infty, R) = (H^\infty, A) = (S_+, L_+) = L_+$.

Proof. The equation $(H^\infty, R) = L_+$ is merely a restatement of Theorem 1. Indeed, the existence of the limit (1) is precisely the existence of the radial limit along the unit interval; the existence of the limit along any other radius follows by rotation.

Clearly, $(H^\infty, A) \subset (H^\infty, R) = L_+$. Conversely, the convolution of a function in $L^1$ with a function in $L^\infty$ is continuous.

The identity sequence $\{1, 1, \cdots\}$ is in $S_+$, and therefore $(S_+, L_+) \subset L_+$. The reverse inclusion follows from the fact that $(S, L) = L$ for two-sided sequences (see Zygmund [8, Chapter IV, Theorem 11.10]). This completes the proof of the theorem.

The third equation in Theorem 3, $(S_+, L_+) = L_+$, is in some sense a dual of the second equation, $(H^\infty, A) = L_+$. As we have just seen, it is easy to prove. However, we do not know how to establish the second equation without going through Theorem 1. Perhaps this is to be expected; for although $S_+$ is the conjugate space of the Banach space $A$, the space $L_+$ only becomes the full dual of $H^\infty$ when $H^\infty$ is given the weak-star topology; as we mentioned in the Introduction, Theorem 1 may be regarded as a statement about weak-star continuous linear functionals on $H^\infty$.

In conclusion, we mention a conjecture that seems to be difficult to settle (it implies Theorem 1).

CONJECTURE. *Let* $\{\phi_n\}$ *be a sequence of elements in* $L$ *such that* $\lim \int \phi_n f$ *exists for each* $f \in H^\infty$. *Then there exists a* $\phi \in L$ *such that*

$$\lim \int \phi_n f = \int \phi f \qquad \text{for all } f \in H^\infty. \tag{13}$$

To state the problem differently: is the quotient Banach space $L/H$ weakly sequentially complete? (This question was raised in [**1**, pp. 180–181].) The analogous result for $L$ instead of $L/H$ (that is, with $L^\infty$ instead of $H^\infty$ in the statement above) was proved by Steinhaus [**6**] (see also the proof in Zygmund [**8**, Chapter IV, Theorem 9.13]).

In the paper immediately following this one, Kahane [**3**] gives a partial affirmative answer to the conjecture. He shows that there exists a $\phi \in L$ such that the relation (13) holds for all $f \in A$. This is enough to imply Theorem 1.

## REFERENCES


1. L. Brown, A. Shields and K. Zeller, *On absolutely convergent exponential sums*, Trans. Amer. Math. Soc. **96** (1960), 162–183.
2. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
3. J.-P. Kahane, *Another theorem on bounded analytic functions and absolutely continuous measures*, Proc. Amer. Math. Soc. **18** (1967), 827–831.
4. *Analytic function theory*, (Sympos. on Analytic Function Theory, Univ. of Kentucky, 1965), Bull. Amer. Math. Soc. **71** (1965), 855–857.
5. L. A. Rubel and A. L. Shields, *The space of bounded holomorphic functions on a region*, Ann. Inst. Fourier (Grenoble) **16** (1966), 235–277.
6. H. Steinhaus, *Additive und stetige Funktionaloperationen*, Math. Z. **5** (1919), 186–221.
7. A. E. Taylor, *Banach spaces of functions analytic in the unit circle*. II, Studia Math. **12** (1951), 25–50.
8. A. Zygmund, *Trigonometric series*, 2nd ed., Vol. I, Cambridge Univ. Press, Cambridge, 1959.


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