ANOTHER THEOREM ON BOUNDED
ANALYTIC FUNCTIONS

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This note is an attempt to solve the conjecture stated at the end of the preceding paper [1]. We are able to prove the following.

Theorem 1. Let \( \{\phi_n\} \) be a sequence of summable functions on the circle such that

\[
\lim_{n \to \infty} \int f \phi_n
\]

exists for all \( f \in H^\infty \) (space of bounded functions on the circle with a positive spectrum; the integral is taken over the circle). Then there is a \( \phi \in L^1 \) such that

\[
l(f) = \int f \phi
\]

for all \( f \in A \) (space of continuous functions on the circle with a positive spectrum).

Proof. As in [1] we see first that there exists a measure \( d\mu \) on the circle such that \( l(f) = \int fd\mu \) whenever \( f \in A \). Let us prove that \( d\mu \) is absolutely continuous.

Suppose that \( d\mu \) is not absolutely continuous. Let \( E \) be a closed set on the circle, with Lebesgue measure zero, such that \( \int_E d\mu = \mu(E) \neq 0 \). Let \( h \) be a function in \( A \) such that \( h = 1 \) on \( E \) and \( |h| < 1 \) outside (the existence of such a function is well known; it is used also in [1]). We have the following equalities (\( m = 1, 2, \ldots; n = 1, 2, \ldots \)):

1. \( \lim_{m \to \infty} \int h^m d\mu = \mu(E) \),
2. \( \lim_{m \to \infty} \int h^m \phi_n = 0 \) for all \( n \)'s,
3. \( \lim_{n \to \infty} \int h^n \phi_n = \int h^m d\mu \) for all \( m \)'s.

If the sequence \( m_j \) is rapidly increasing (meaning that \( m_{j+1} \) is sufficiently large when \( m_j \) is given), we have

\[
f = \sum_{j=1}^{\infty} (-1)^j h^{m_j} \in H^\infty.
\]

For, given \( m_j \), we define \( E_j \) as the set where \( |h^{m_j} - 1| < 2^{-j} \), and we have \( |h^{m_{j+1}}| < 2^{-j} \) on \( CE_j \) when \( m_{j+1} \) is large enough. We shall write \( L_1 \) for this condition on the \( m_j \).

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We shall define by induction two sequences \( m_j \) (satisfying \( L_1 \)) and \( n_j \). We shall use the formula

\[
\int f \phi_{n_j} = \sum_{k=1}^{j-1} (-1)^k \int h^{m_k} \phi_{n_j} + (-1)^j \int h^{m_j} \phi_{n_j} + \sum_{k=j+1}^{\infty} (-1)^k \int h^{m_k} \phi_{n_j},
\]

\[
= A_j + B_j + C_j.
\]

We write \( L_2 \) for the condition

\[
\sum_{k=j+1}^{\infty} \left| \int h^{m_k} \phi_{n_j} \right| < \frac{1}{12} | \mu(E) | ;
\]

by (2), it is satisfied when \( m_{j+1}, m_{j+2}, \ldots \) are chosen large enough, \( n_j \) being given. We write \( L_3 \) for the condition

\[
\left| \int h^{m_j} d\mu \right| > \frac{11}{12} | \mu(E) | ;
\]

by (1), it is satisfied when \( m_j \) is large. Now suppose that \( m_1, \ldots, m_{j-1}, n_1, \ldots, n_{j-1} \) are given in such a manner that the conditions \( L_1, L_2, L_3 \) are satisfied at this stage. They will be satisfied at the following stage if \( m_j \) is sufficiently large, \( m_j \geq m^*_j \), say. We define \( n^*_j \) so that \( n \geq n^*_j \) implies

\[
| A_j - A^\infty_j | < \mu(E) / 12,
\]

where

\[
A^\infty_j = \sum_{k=1}^{j-1} (-1)^k \int h^{m_k} d\mu;
\]

that is possible because of (3). Now we consider two cases, namely

(\( \alpha \)) \( | A_{j-1} + B_{j-1} - A^\infty_j | \leq 5 | \mu(E) | / 12, \)

(\( \beta \)) \( | A_{j-1} + B_{j-1} - A^\infty_j | > 5 | \mu(E) | / 12. \)

In the case (\( \alpha \)), we choose \( m_j = m^*_j \), and \( n_j \) large enough \( (\geq n^*_j) \) so that \( | B_j | > 11 | \mu(E) | / 12 \); that is possible because of (3) and \( L_3 \). In the case (\( \beta \)), we choose \( n_j = n^*_j \), and \( m_j \) large enough \( (\geq m^*_j) \) so that \( | B_j | < | \mu(E) | / 12 \); that is possible because of (2). In each case, we have

\[
| A_{j-1} + B_{j-1} - A_j - B_j | > 3 | \mu(E) | / 12.
\]

Taking \( L_2 \) into account, we have \( | C_{j-1} | \) and \( | C_j | \) majorized by \( | \mu(E) | / 12 \), and therefore

\[
\left| \int f \phi_{n_{j-1}} - \int f \phi_{n_j} \right| > \frac{1}{12} | \mu(E) | .
\]
Therefore the sequence $\int f \phi_n$ is not convergent, against our assumption. The contradiction proves that $d\mu$ is absolutely continuous, that is $l(f) = \int f d\mu = \int f \phi$ whenever $f \in A$, for some $\phi \in L^1$.

**Remark.** If $\phi_n(t) = \sum_{k=-\infty}^{\infty} a_{n,k} e^{-ikt}$, the assumption of the theorem is the existence of $\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k} b_k$ for all $\sum_{k=0}^{\infty} b_k e^{ikt} \in H^\infty$. The conclusion is $\lim_{n \to \infty} a_{n,k} = \int \phi(\theta) e^{ikt}$ for some $\phi \in L^1$ ($k = 0, 1, 2, \cdots$). Theorem 1 of [1] follows as a particular case.

We are not able to prove that $l(f) = \int f \phi$ for all $f \in H^\infty$. Nevertheless, this holds for many functions in $H^\infty$. Precisely, we have

**Theorem 2.** Keeping the same notations as in Theorem 1, let $D_1$ be the set of all $f \in H^\infty$ such that $l(f) = \int f \phi$, and let $D$ be the intersection of the $D_1$ for all $l$. Then (a) $D_1$ is a closed subspace of $H^\infty$ and, given any $f \in H^\infty$, almost all translates of $f$ belong to $D_1$. (b) $D$ is a closed subalgebra of $H^\infty$, invariant under translation; it contains all $f \in H^\infty$ such that $fg \in D$ for some outer function $g \in D$; in particular, it contains all $f \in H^\infty$ which are continuous on the circle except on a closed set of measure zero.

**Proof.** We may suppose that the $\phi_n$ are trigonometric polynomials. By the Banach-Steinhaus theorem, the linear functionals $f \mapsto \int f \phi_n$ are uniformly bounded on $A$. There exist measures $d\mu_n$, with bounded norms, such that $\int f \phi_n = \int f \, d\mu_n$ for all $f \in A$. By the F. and M. Riesz theorem (or another device) the $d\mu_n$ are absolutely continuous. Therefore we may suppose that the $\phi_n$ have bounded $L^1$-norms.

In order to prove (a) we may suppose $\phi = 0$. The fact that $D_1$ is a closed subspace of $H^\infty$ is obvious. Given $f \in H^\infty$, we write $f_\ast(t) = f(t - s)$. Given $\psi \in L^1$, we have $f_\ast \psi \in A$, and by Theorem 1

$$\lim_{n \to \infty} \int \int \phi_n(t) f(t - s) \psi(s) \, ds \, dt = 0.$$ 

By assumption

$$\lim_{n \to \infty} \int \phi_n(t) f(t - s) \, dt = l(f_\ast)$$

and since the $\phi_n$ have bounded $L^1$-norms, the integrals $\int \phi_n(t) f(t - s) \, dt$ are uniformly bounded with respect to $n$ and $s$. By the Lebesgue theorem

$$\int l(f_\ast) \psi(s) \, ds = 0$$

and since $\psi$ is an arbitrary function in $L^1$, $l(f_\ast) = 0$ for almost every $s$. That proves (a).
In order to prove (β) we write
\[
\lim_{n \to \infty} \int f g \phi_n = l_f(g) = l_\phi(f) = l(fg) \quad (f \in H^\infty, g \in H^\infty),
\]
\[
l_f(g) = \int g \phi_f \quad \text{when } g \in D.
\]

We have $A \subseteq D$ as a reformulation of Theorem 1.
Suppose $f \in A$. Taking $g \in A$, we have $fg \in A$. Since $fg \in D$ and $g \in D$, we have
\[
\int f g \phi = l(fg) = l_f(g) = \int g \phi_f,
\]
and since $g$ is arbitrary in $A$, $f \phi = \phi_f$ (mod $H_0^\infty$) (meaning that the Fourier coefficients or order $\leq 0$ are the same).

Now suppose $g \in D$. Taking $f \in A$ we have
\[
\int g \phi = l(g) = \int g \phi_f = \int f g \phi
\]
since $f \phi = \phi_f$ (mod $H_0^\infty$). Therefore $fg \in D_1$ and, $l$ being arbitrary, $fg \in D$. Since $fg \in D$ and $g \in D$, we have
\[
\int f(g \phi - \phi_\phi) = 0
\]
and since $f$ is arbitrary in $A$, $g \phi = \phi_\phi$ (mod $H_0^\infty$).

If $f \in D$ and $g \in D$, we still have (6) because $f \phi = \phi_f$ (mod $H_0^\infty$), and $fg \in D$ as a consequence. Therefore $D$ is a subalgebra of $H^\infty$. It is closed because each $D_1$ is closed, and it is obviously invariant under translation.

Finally, suppose that $f \in H^\infty$, $g \in D$, $g$ is an outer function and $fg \in D$. We still have (4) and (5). Moreover, since $D$ is an algebra, we have
\[
\int g h(f \phi - \phi_f) = 0
\]
for all $h \in D$. Therefore $g(f \phi - \phi_f) = 0$ (mod $H_0^\infty$). Since $g$ is an outer function, it follows that $f \phi = \phi_f$ (mod $H_0^\infty$). As a conclusion
that is, $f \in D_I$, and since $l$ is arbitrary, $f \in D$.

Given a closed set $K$ of measure zero on the circle, there exists a continuous outer function $g$ vanishing on $K$ (that follows immediately from a proof of Fatou's theorem). If $f$ is continuous except on $K$, $fg \in A$, therefore $f \in D$. That ends the proof of Theorem 2.

**Reference**