A MAXIMALITY THEOREM IN FOURIER ANALYSIS

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Wermer's well-known maximality theorem [4], [5] has been generalized by Hoffman and Singer [2] as follows. Let G be an ordered abelian group with nonnegative class G_+, and dual group Γ. Then the subalgebra A of $C(\Gamma)$ generated by G_+ can be extended to a maximal subalgebra of $C(\Gamma)$, provided there exists a homomorphism $\xi \neq 0$ of G into the additive group R with $\xi \leqq 0$ on G_+. The homomorphism ξ is unique to within multiplication by a positive scalar, and the maximal subalgebras so obtained are translates of each other, as set forth precisely in [2]. (See [1, pp. 193–194] for an interesting investigation of maximality. A is a Dirichlet algebra, [1].)

Our theorem is a converse: suppose that B is a maximal proper closed subalgebra of $C(\Gamma)$, or of $l_1(G)$, and that for each element g of G, either $g \in B$ or $g^{-1} \in B$. Of course, $l_1(G)$ is construed as a (dense) subalgebra of $C(\Gamma)$ *via* the Fourier transform.

THEOREM. *There is a complex homomorphism Φ of B with the property that $0 < |\Phi| \leqq 1$ on $G \cap B$ and $|\Phi|$ is not identically 1 on $G \cap B$. In particular, if A is contained in a maximal subalgebra of $C(\Gamma)$, the mapping $\log |\Phi|$ of $G_+ \cap B$ can be extended to a negative mapping ξ of G into R.*

PROOF. Let $g_1, g_2 \in G \cap B$, $g_2^{-1} \notin B$. We assert that for some $n \geqq 1$, $g_2^n g_1^{-1} \in B$. For otherwise, $g_2^{-n} g_1 \in B$ for all $n \geqq 1$ and $g_2^{-n} g_1$ is singular, so the distance $\|1 - g_2^{-n} g_1 B\| = 1$. Since the multiplications by G are isometries of $l_1(G)$ and $C(\Gamma)$, $\|g_2^n - g_1 B\| \geqq 1$, for $n \geqq 1$. Applying the spectral radius formula to the algebra $B/g_1 B$ we find a complex homomorphism Ψ of B for which $\Psi(g_1) = 0$, $|\Psi(g_2)| = 1$.

The homomorphism Ψ admits an extension to the subalgebra generated by B and g_2^{-1}, according to the rule $g_2^m x \to \Psi(g_2)^m \Psi(x)$, $x \in B$, $-\infty < m < \infty$. To see that this is bounded (and well-defined) note that

$$\left| \sum_{-N}^{N} \Psi(g_2)^m \Psi(x_m) \right| = \left| \sum_{-N}^{N} \Psi(g_2)^{m+N} \Psi(x_m) \right| \leqq \left\| \sum_{-N}^{N} g_2^{m+N} x_m \right\| = \left\| \sum_{-N}^{N} g_2^m x_m \right\|.$$

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The algebra in question is larger than B because it contains g_2^{-1}, and not dense because g_1 is singular in its closure. This contradiction proves our assertion.

Finally, let g be any singular element of $G\cap B$. The spectrum of g in $l_1(G)$, or $C(\Gamma)$, is just the unit circle, so that if $\lambda \in \mathrm{Sp}_B g$ and $|\lambda| < 1$, then λ is an interior point of $\mathrm{Sp}_B g$, by spectral permanence [**3**, pp. 33, 142]. Since $0 \in \mathrm{Sp}_B g$ this last set contains the entire unit disk. Then there is a homomorphism Φ of B with $0 < |\Phi(g)| < 1$. Our assertion in the first paragraph shows that $0 < |\Phi(g_1)| \leqq 1$ for all $g_1 \in G\cap B$, because for some $n \geqq 1$, $g^n \in g_1 B$. End of proof.

The weak direct sum $\cdots \oplus Z \oplus Z \oplus Z$ can be ordered lexicographically, and admits only the trivial positive mapping into R. The algebra $A(\Gamma)$ is then included in no maximal subalgebra. The Gelfand space of $A(\Gamma)$ seems curious enough to reward a careful investigation.

References

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