CONTINUOUS EXTENSIONS

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1. **Introduction.** Dugundji [2] proved the remarkable result that *every continuous function f from a closed subset A of a metric space X to a locally convex linear topological space admits a continuous extension over X*, and in [1] we proved that this result remains valid whenever X is a stratifiable[2] space. The proofs of these results essentially depend on forming certain convex combinations of the values of f with barycentric coordinates attached to points of $X-A$ in a continuous fashion. Naturally one wonders if the above results can be significantly improved if

(a) There exists an upper bound for the size (i.e., the number of nonzero barycentric coordinates) of the convex combinations involved in the proof of the above results.

(b) The function f has extra properties (for example, f is closed, or f is linear).

While we do not have the complete answer to these questions, we will, however, prove, among other results, the following:

(A) Every continuous function f from a closed subset of a finite-dimensional (in the covering sense) stratifiable space X to a linear topological space L admits a continuous extension (see Theorem 3.1).

(B) Every continuous linear function T from a convex subset of a stratifiable locally convex linear topological space X to a linear topological space L admits a (not necessarily linear) continuous extension (indeed, we will prove much stronger results).

Both of the preceding results seem to be new even when X is n-Euclidean space. The proof of (A) requires that we prove some results about the structure of a linear topological space (see Theorem 2.2) which may hold independent interest.

2. **Preliminary results.** Before we can prove statement (A) in the introduction, we need to show that every linear topological space is *not far from being locally convex* in the sense described below:

DEFINITION 2.1. Let L be a linear topological space. Then L is said

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[2] A T_1-topological space X is stratifiable if to each open $U\subset X$ one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of open subsets of X such that, for all n, $U_n\subset V_n$ whenever $U\subset V$, $U_n^-\subset U$ and $\bigcup_{n=1}^{\infty}U_n=U$. The correspondence $U\rightarrow\{U_n\}_{n=1}^{\infty}$ is called a *stratification* of X. We note that metrizability implies stratifiability which implies paracompact and perfectly normal, and all CW-complexes of Whitehead are stratifiable.

to be *locally n-convex*, where n is a positive integer, provided that, for each $x \in L$ and neighborhood U of x, there exists a neighborhood V of x such that $\text{conv}_n V \subset U$, where

$$\text{conv}_n V = \left\{ \sum_{i=1}^{n} t_i v_i \,\middle|\, \sum_{i=1}^{n} t_i = 1, v_i \in V, 0 \leqq t_i \leqq 1 \text{ for each } i \right\}$$

THEOREM 2.2. *Every linear topological space L is locally n-convex, for each positive integer n.*

PROOF. Define Φ: $L \times L \times I \to L$ by $\Phi(x, y, t) = tx + (1-t)y$; this is continuous because L is a linear space. Given $w \in L$ and some neighborhood U of w, the set $\Phi^{-1}(U)$ is therefore open; since $w \times w \times I \subset \Phi^{-1}(U)$, there is [**3**, p. 228 (2.6)] a neighborhood V of w such that $\Phi(V, V, I) \subset U$ and therefore $\text{conv}_2 V \subset U$. It is trivial to see that $\text{conv}_n U \subset V$ and $\text{conv}_2 V \subset W$ implies $\text{conv}_{n+1} U \subset W$ $(n \geqq 2)$ so the theorem follows by induction.

REMARK 2.3. A linear space with the weak topology with respect to its finite-dimensional Euclidean subspaces may not even be locally 2-convex (see the proof of Theorem 4.3 on p. 416 of Dugundji [**3**]).

3. **Theorems and proofs.** Throughout this section, let $F(Y, Z; P)$ be the space of functions from Y to Z satisfying P, for any topological spaces Y and Z and property P.

THEOREM 3.1. *Let X be a stratifiable space, A a closed subset of X, L any linear topological space. If* $\dim(X-A)$ *is finite, then there exists a mapping*

$$\Phi: F(A, L; \textit{continuous}) \to F(X, L; \textit{continuous})$$

satisfying the following:

(a) $\Phi(f)$ *is an extension of* f,

(b) *the range of* $\Phi(f)$ *is contained in the convex hull of the range of* f,

(c) Φ *is a linear transformation*,

(d) Φ *is a continuous whenever both function spaces have the compact-open topology, the topology of pointwise convergence, or the topology of uniform convergence.*

PROOF. The proof of this and the following results is very similar to the proof of Theorem 4.3 in [**1**] together with the paragraph which follows it. However, it is best that we indicate the general procedure, without details, in a way that will help us in latter proofs. We briefly comment on some *recurring notation throughout ensuing proof*: Letting $U \to \{U_n\}_{n=1}^{\infty}$ be a stratification of X (see footnote 2), we let

$n(U, x)=\min\{n \mid x\in U_n\}$ and $U_x=U_{n(U,x)}-(X-\{x\})^-_{n(U,x)}$, for each open $U\subset X$ and $x\in U$. It is easily seen (see Lemma 4.2 of [1]) that each U_x is an open neighborhood of x, $U_x\cap V_y\neq\Phi$ implies $x\in V$ or $y\in U$ (indeed, $n(U, x)\leqq n(V, y)$ implies $y\in U$).

Let $W=X-A$, $W'=\{x\in W \mid w\in U_y$ for some $y\in A$ and open U containing $y\}$ and $m(x)=\max\{n(u, y) \mid y\in A$ and $x\in U_y\}$, for each $x\in W'$. It is easily seen that $m(x)<n(W, x)<\infty$.

Since W is finite-dimensional and paracompact there exists a locally finite open refinement $\mathcal{V}$ of $\{W_x \mid x\in W\}$ (with respect to W) such that (see, for example, Theorem 2.6 of Nagata [7]).

(A) At most $n+1$ elements of $\mathcal{V}$ have a nonempty intersection for some positive integer n.

Let $\{p_V \mid V\in\mathcal{V}\}$ be a partition of unity subordinated to $\mathcal{V}$ and define $g: X\to E$ by

$$g(x) = f(x) \qquad \text{if } x \in A,$$

$$g(x) = \sum_{V\in\mathcal{V}} p_V(x)f(a_V) \qquad \text{if } x \in W,$$

where the a_V are chosen as follows: For each $V\in\mathcal{V}$ pick $x_V\in W$ with $V\in W_{x_V}$. If $x_V\in W'$ pick $a_V\in A$ and open S_V containing a_V such that $x_V\in(S_V)_{a_V}$ and $n(S_V, a_V)=m(x_V)$; if $x_V\notin W'$, let a_V be the fixed point $a_0\in A$.

Clearly $g(X)\subset\operatorname{conv}_{n+1}f(A)\subset$convex hull of $f(A)$ and g is continuous on W. To show that g is continuous at $b\in A$, let 0 and K be any open subsets of L containing $f(b)$ such that (by Theorem 2.2)

(B) $\operatorname{conv}_{n+1} K \subset 0$,

and let N be an open neighborhood of b such that

(C) $f(A)\cap N)\subset K$.

It is then easily seen that

(D) $x\in(N_b)_b-A$ implies $a_V\in N$ for each $V\in\mathcal{V}$ with $x\in V$.

Hence $g((N_b)_b)\subset\operatorname{conv}_{n+1}K\subset 0$ and thus g is continuous. The function Φ defined by $\Phi(f)=g$ is easily seen to have the required properties.

Theorem 3.1 is true if one replaces "dim" by "Ind" since Vedenisov [8] has shown that dim $X\leqq$ Ind X for normal spaces.

Since *every topologized vector space is* clearly *locally* 1-*convex*, Theorem 3.1 remains true if X is any stratifiable space with $\dim(X-A)=0$ and L is any topologized vector space.

Now we turn our attention to the extension of linear functions.

DEFINITION 3.4. Let X and Y be convex spaces (i.e., convex combinations of points of these spaces are well-defined points of these spaces) and $f: X \to Y$ be a function. Then f is said to be a *convex function* provided that

$$f\left(\sum_{i=1}^{n} t_i x_i\right) = \sum_{i=1}^{n} t_i f(x_i)$$

for each convex combination $\sum_{i=1}^{n} t_i x_i$ of elements of X.

Clearly, *every linear function is a convex function.*

THEOREM 3.5. *Let E be a* (*convex subset of a*) *stratifiable locally convex linear topological space, A a convex closed subset of E, L any topologized vector space. Then there exists a mapping*

$$\Phi: F(A, L; \text{convex continuous}) \to F(E, L; \text{convex continuous})$$

satisfying conditions (a)–(d) *of Theorem* 3.1 *and*

$$\text{range } f = \text{range } \Phi(f) \text{ for each } f.$$

PROOF. Essentially the same as the proof of Theorem 3.1 except that we remove statement (A) and replace statements (B) and (C), respectively, by

(B′) $K \subset 0$,

(C′) $f(A \cap N) \subset K$ and N is convex.

It will then follow from statement (D) that $g((N_b)_b) \subset 0$, because f is a convex function.

It is worth noting that a *continuous linear function f from a subspace of a Banach space B_1 to another Banach space B_2 may not even have a bounded linear extension* (see Lindenstrauss [6]); however, our Theorem 3.5 clearly implies that f *has a norm-preserving continuous extension.*

Theorem 3.5 is true if L is any full[3] simplicial complex; indeed, in this case, we can also replace E by any full complex with the metric topology since, according to the proof of Theorem 26.2 of Hanner [5], E is isometric with a convex subspace of a Banach space.

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[3] A simplicial complex C is called *full* if for every finite set $\{v_1, \cdots, v_n\}$ of vertices of C the simplex $\langle v_1, \cdots, v_n\rangle$ is contained in C.

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