A VARIANT OF TCHEBICHEF'S MINIMAX PROBLEM

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1. **Introduction.** Let Q_k be the set of all monic polynomials of degree $k>0$ with real coefficients. Let Σ_h be the set of all those subsets of the interval $[-1, +1]$ which contain -1 and $+1$ and have at least one point in each closed subinterval of length h. What is the minimum value of $\max_{x\in\sigma}|q(x)|$ as q ranges over Q_k while σ ranges over Σ_h? When $h=0$, the only member of Σ_h is the entire interval $[-1, +1]$, so that in this case the problem has the classical solution due to Tchebichef,

$$\max_{x\in\sigma} |q(x)| \geqq \max_{-1\leqq x\leqq 1} |\tilde{T}_k(x)| = 2^{1-k}, \tag{1-1}$$

where $\tilde{T}_k(x)$ is the monic Tchebichef polynomial, $\tilde{T}_k(x) = 2^{1-k}\cdot$ $\cos(k \arccos x)$. (See, for example, Natanson [3, Chap. II].) When $k=1$, $(1-1)$ holds for every $\sigma\in\Sigma_h$ independently of the value of h; so we henceforth take $k>1$. When $h\geqq 2/(k-1)$, the problem is trivial, since $\max_{x\in\sigma}|q(x)|$ attains zero as its minimum value for $\sigma=\{x_0, x_1, \cdots, x_{k-1}\}$, $q(x)=(x-x_0)(x-x_1)\cdots(x-x_{k-1})$, where $x_i=-1+2i/(k-1)$. There remains the case, $0<h<2/(k-1)$; and the object of this paper is to construct for any h in this range a polynomial $T_k^{(h)}(x)$ in Q_k and a subset σ^* in Σ_h which together minimize $\max_{x\in\sigma}|q(x)|$ on the product space $Q_k\times\Sigma_h$. This result is obtained in Theorem 4. The corollary is used in another paper [2] to derive an irreducibility criterion for polynomials over the rational field. The polynomials $T_k^{(h)}(x)$ are defined and their relevant properties established in §3. §2 is a preliminary section in which are developed the required properties of certain Hahn polynomials involved in the construction of $T_k^{(h)}(x)$.

2. **Certain Hahn polynomials.** The solution to our problem depends upon the properties of the Hahn polynomials $P_m^{(\alpha,\beta,\gamma)}(x)$ with the special parameter values $\alpha=\pm 1/2$, $\beta=\pm 1/2$. (All properties of the Hahn polynomials needed here as well as references to other literature on these polynomials can be found in the author's paper [1].) To simplify the notation we define

$$R_m(x) = R_m^{(h)}(x) = \frac{(2h)^m m!}{(2m)!} P_m^{(-1/2,1/2,1/h)}\left(\frac{x+1}{2h}\right), \tag{2-1}$$

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$$(2\text{-}2)\qquad S_m(x) = S_m^{(h)}(x) = \frac{(2h)^m m!}{(2m)!} P_m^{(1/2,-1/2,1/h)}\left(\frac{x+1-h}{2h}\right),$$

$$(2\text{-}3)\qquad U_m(x) = U_m^{(h)}(x) = \frac{(2h)^m(m+1)!}{(2m+1)!} P_m^{(1/2,-1/2,1/h-1/2)}\left(\frac{x+1-h}{2h}\right),$$

$$(2\text{-}4)\qquad V_m(x) = V_m^{(h)}(x) = \frac{(2h)^m(m-1)!}{(2m-1)!} P_m^{(-1/2,-1/2,1/h+1/2)}\left(\frac{x+1}{2h}\right),$$

with the understanding that $R_0(x) = S_0(x) = U_0(x) = V_0(x) = 1$. These are all monic polynomials of degree m with real coefficients. Each of the following formulas can be readily obtained by specializing the indicated formula from [1].

$$(2\text{-}5)\qquad \begin{aligned}&(x+h-1)R_m(x+h) - (x-1)R_m(x-h)\\ &\quad = (2m+1)hS_m(x),\end{aligned} \qquad [1, (2.7)],$$

$$(2\text{-}6)\qquad \begin{aligned}&(x+h+1)S_m(x+h) - (x+1)S_m(x-h)\\ &\quad = (2m+1)hR_m(x),\end{aligned} \qquad [1, (2.3)],$$

$$(2\text{-}7)\qquad \begin{aligned}&(x+h+1)(x+h-1)U_{m-1}(x+h)\\ &\quad - (x+1)(x-1)U_{m-1}(x-h) = 2mhV_m(x),\end{aligned} \qquad [1, (4.3)],$$

$$(2\text{-}8)\qquad V_m(x+h) - V_m(x-h) = 2mhU_{m-1}(x), \qquad [1, (4.1)],$$

$$(2\text{-}9)\qquad R_m(-1) = (-2)^{-m}\prod_{i=1}^{m}(1-ih), \qquad [1, (3.0)],$$

$$(2\text{-}10)\qquad V_m(-1) = 2(-2)^{-m}\prod_{i=1}^{m}\left[1-\left(i-\tfrac{1}{2}\right)h\right], \qquad [1, (3.0)],$$

$$(2\text{-}11)\qquad R_m(-1-h) = (-2)^{-m}\prod_{i=1}^{m}(1+ih), \qquad [1, (3.4)],$$

$$(2\text{-}12)\qquad V_m(-1-h) = 2(-2)^{-m}\prod_{i=1}^{m}\left[1+\left(i-\tfrac{1}{2}\right)h\right], \qquad [1, (3.4)].$$

(2-5) and (2-6) hold for $m \geqq 0$, the rest for $m \geqq 1$.

Next, we establish the formulas,

$$(2\text{-}13)\qquad \begin{aligned}&(x-1)R_m(x)R_m(x-h) + C_{2m+1}\\ &\quad = (x+1)S_m(x)S_m(x-h) - C_{2m+1},\end{aligned} \qquad m \geqq 0,$$

where

$$(2\text{-}13a)\qquad C_1 = 1, \quad C_{2m+1} = C_{2m+1}(h) = 2^{-2m}\prod_{i=1}^{m}(1-i^2h^2), \qquad m \geqq 1,$$

$$(2\text{-}14)\quad \begin{aligned}(x^2-1)U_{m-1}(x)U_{m-1}(x-h)+C_{2m}\\ = V_m(x)V_m(x-h)-C_{2m},\end{aligned}\qquad m \geqq 1,$$

where

$$(2\text{-}14a)\quad C_{2m} = C_{2m}(h) = 2^{1-2m}\prod_{i=1}^{m}\left[1-(i-\tfrac{1}{2})^2h^2\right],\qquad m\geqq 1.$$

To prove (2-13) we multiply (2-5) through by $R_m(x)$ and (2-6) by $S_m(x)$ and find that

$$(2\text{-}15.1)\quad \begin{aligned}\Delta_h[(x-1)R_m(x)R_m(x-h)] &= (2m+1)hR_m(x)S_m(x),\\ \Delta_h[(x+1)S_m(x)S_m(x-h)] &= (2m+1)hR_m(x)S_m(x),\end{aligned}$$

where $\Delta_h f(x) = f(x+h)-f(x)$. The bracketed expressions in the left members are polynomials in x, which thus have equal first differences with respect to x. Hence, we conclude that

$$(x+1)S_m(x)S_m(x-h)-(x-1)R_m(x)R_m(x-h) = A,$$

where A is independent of x. To evaluate A, we set $x=-1$ and find that

$$A = 2R_m(-1)R_m(-1-h) = 2^{1-2m}\prod_{i=1}^{m}(1-i^2h^2),\qquad m\geqq 1,$$

by (2-9), (2-11), which yields (2-13). (2-14) is derived similarly by using (2-7), (2-8) to obtain

$$(2\text{-}15.2)\quad \begin{aligned}\Delta_h[(x^2-1)U_{m-1}(x)U_{m-1}(x-h)] &= 2mhU_{m-1}(x)V_m(x),\\ \Delta_h[V_m(x)V_m(x-h)] &= 2mhU_{m-1}(x)V_m(x),\end{aligned}$$

and then applying (2-10), (2-12).

Next we consider the location and interlacing of the zeros of these polynomials.

THEOREM 2-1. *Let* $0<h<1/m$, $m\geqq 1$. *Then* $R_m^{(h)}(x)$ *and* $S_m^{(h)}(x)$ *have all real and simple zeros,* $r_1<r_2<\cdots<r_m$ *and* $s_1<s_2<\cdots<s_m$ *respectively, which are located and spaced as follows:*

$$(2\text{-}16)\quad -1<r_i<1-2h,\qquad -1+h<s_i<1-h\qquad (i=1,2,\cdots,m);$$

$$(2\text{-}17)\quad r_{i+1}-r_i>2h,\qquad s_{i+1}-s_i>2h\qquad (i=1,2,\cdots,m-1),\ m>1.$$

The zeros of $R_m^{(h)}(x)$ *and* $S_m^{(h)}(x)$ *separate each other. In fact,*

$$(2\text{-}18)\qquad \begin{aligned} s_i - r_i &> h \qquad (i = 1, 2, \cdots, m); \\ r_{i+1} - s_i &> h \qquad (i = 1, 2, \cdots, m-1),\ m > 1. \end{aligned}$$

PROOF. If $\gamma > m$, the zeros of $P_m^{(\alpha,\beta,\gamma)}(x)$ are all real and simple and contained in the open interval $(0, \gamma - 1)$ [1, Theorem 1]. As applied to $R_m(x)$ and $S_m(x)$ this yields (2-16). (2-17) follows from the fact that consecutive zeros of $P_m^{(\alpha,\beta,\gamma)}(x)$ are more than one unit apart [1, Theorem 3]. Finally, let $a_1 < a_2 < \cdots < a_m$ be the zeros of $P_m^{(\alpha,\beta,\gamma)}(x)$ and $b_1 < b_2 < \cdots < b_m$ be the zeros of $P_m^{(\alpha-1,\beta+1,\gamma)}(x)$. Then, $a_i > b_i$ $(i = 1, 2, \cdots, m)$, and $b_{i+1} > a_i + 1$ $(i = 1, 2, \cdots, m-1)$ [1, Theorem 4]. Setting $\alpha = 1/2$, $\beta = -1/2$, $\gamma = 1/h$, we have (2-18).

THEOREM 2-2. *Let $0 < h < 2/(2m-1)$. Then $V_m^{(h)}(x)$ for $m \geqq 1$ and $U_{m-1}^{(h)}(x)$ for $m > 1$ have all real and simple zeros, $v_1 < v_2 < \cdots < v_m$ and $u_1 < u_2 < \cdots < u_{m-1}$ respectively, which are located and spaced as follows:*

$$(2\text{-}19)\qquad \begin{aligned} -1 + h < u_i &< 1 - 2h \qquad (i = 1, 2, \cdots, m-1), \\ -1 < v_i &< 1 - h \qquad (i = 1, 2, \cdots, m); \end{aligned}$$

$$(2\text{-}20)\qquad \begin{aligned} u_{i+1} - u_i &> 2h \qquad (i = 1, 2, \cdots, m-2),\ m > 2; \\ v_{i+1} - v_i &> 2h \qquad (i = 1, 2, \cdots, m-1),\ m > 1. \end{aligned}$$

The zeros of $U_{m-1}^{(h)}(x)$ and $V_m^{(h)}(x)$ separate each other. In fact,

$$(2\text{-}21)\qquad u_i - v_i > h, \quad v_{i+1} - u_i > h \qquad (i = 1, 2, \cdots, m-1),\ m > 1.$$

The proof is the same as in the preceding theorem except that in establishing (2-21) we utilize the result that there is precisely one zero of $P_m^{(\alpha+1,\beta+1,\gamma-1)}(x)$ in each interval $(a_i, a_{i+1} - 1)$ [1, Theorem 7].

3. **The polynomials $T_k^{(h)}(x)$.** We define the polynomials $T_k(x) = T_k^{(h)}(x)$ as follows for k odd and even respectively:

$$(3\text{-}1)\qquad T_{2m+1}^{(h)}(x) = (x+1)S_m^{(h)}(x)S_m^{(h)}(x-h) - C_{2m+1}(h) \qquad (m \geqq 0),$$

$$(3\text{-}2)\qquad T_{2m}^{(h)}(x) = V_m^{(h)}(x)V_m^{(h)}(x-h) - C_{2m}(h) \qquad (m \geqq 1),$$

where $C_k(h)$ is defined by (2-13a) and (2-14a). $T_k(x)$ is evidently a monic polynomial in x of degree $k > 0$ with real coefficients. From (2-13) and (2-14) respectively we have at once

$$(3\text{-}3)\qquad T_{2m+1}(x) = (x-1)R_m(x)R_m(x-h) + C_{2m+1} \qquad (m \geqq 0),$$

$$(3\text{-}4)\qquad T_{2m}(x) = (x^2-1)U_{m-1}(x)U_{m-1}(x-h) + C_{2m} \qquad (m \geqq 1).$$

Next we prove the following pair of theorems.

THEOREM 3-1. *Let* $0<h<1/m$, $m\geqq 1$. *In the notation of Theorem* 2-1,

$$T_{2m+1}(x) \geqq C_{2m+1} \tag{3-5a}$$

when and only when

$$x \geqq 1 \quad \text{or} \quad r_i \leqq x \leqq r_i + h \qquad \text{for any } i. \tag{3-5b}$$

Also,

$$T_{2m+1}(x) \leqq -C_{2m+1} \tag{3-6a}$$

when and only when

$$x \leqq -1 \quad \text{or} \quad s_i \leqq x \leqq s_i + h \qquad \text{for any } i. \tag{3-6b}$$

THEOREM 3-2. *Let* $0<h<2/(2m-1)$, $m\geqq 1$. *In the notation of Theorem* 2-2,

$$T_{2m}(x) \geqq C_{2m} \tag{3-7a}$$

when and only when

$$|x| \geqq 1 \quad \text{or (when } m>1) \quad u_i \leqq x \leqq u_i + h \qquad \text{for any } i. \tag{3-7b}$$

Also,

$$T_{2m}(x) \leqq -C_{2m} \tag{3-8a}$$

when and only when

$$v_i \leqq x \leqq v_i + h \qquad \text{for any } i. \tag{3-8b}$$

Equality holds in (3-5a) if and only if it holds in (3-5b), and similarly for (3-6), (3-7), (3-8). We give the proof for (3-7). The other three cases do not differ significantly. From (3-4) it is clear that $T_{2m}(x) = C_{2m}$ when $x = \pm 1$ and when $x = u_i$ and $x = u_i + h$ $(i=1, 2, \cdots, m-1)$. Let $x_1>1$. In view of (2-19), there is no zero of $U_{m-1}(x)$ in the interval, $x_1-h\leqq x\leqq x_1$. Consequently, $U_{m-1}(x_1-h)$ and $U_{m-1}(x_1)$ have the same sign, and so, by (3-4), $T_{2m}(x_1)>C_{2m}$. Similarly, for a point $x_2<-1$, $T_{2m}(x_2)>C_{2m}$. Next, consider a point x_3 such that for some zero u_i of $U_{m-1}(x)$,

$$u_i \leqq x_3 \leqq u_i + h. \tag{3-9}$$

Then by (2-19), $-1<x_3<1$, so that

$$x_3^2 - 1 < 0. \tag{3-10}$$

By (3-9), $U_{m-1}(x)$ has the zero u_i in the interval $[x_3-h, x_3]$. By (2-20), this is the only zero in that interval. Hence, $U_{m-1}(x_3-h)$ and $U_{m-1}(x_3)$

have opposite signs or one of them vanishes. Combining this fact with (3-10), we conclude that $T_{2m}(x_3) \geqq C_{2m}$ also. Conversely, suppose that $T_{2m}(x_0) \geqq C_{2m}$ for some point x_0 inside $(-1, 1)$. Then by (3-4), $U_{m-1}(x_0)$ and $U_{m-1}(x_0-h)$ must have opposite signs or one of them must vanish. Hence, $U_{m-1}(x)$ has a zero u_i such that $x_0-h \leqq u_i \leqq x_0$. But then, $u_i \leqq x_0 \leqq u_i+h$, which completes the proof.

Next, we note that by (3-1), (2-15.1) and by (3-2), (2-15.2),

$$\begin{aligned} \Delta_h T_{2m+1}(x) &= (2m+1)hR_m(x)S_m(x), \\ \Delta_h T_{2m}(x) &= 2mhU_{m-1}(x)V_m(x). \end{aligned} \tag{3-11}$$

From the second of these we find by (2-7), (2-8) and (3-2), (3-4) that

$$\begin{aligned} &(x+h+1)(x+h-1)\Delta_h T_{2m}(x+h) \\ &\qquad = 4m^2h^2\Delta_h T_{2m}(x) + (x^2-1)\Delta_h T_{2m}(x-h) + 8m^2h^2T_{2m}(x). \end{aligned}$$

This establishes the difference equation,

$$\begin{aligned} &(x+2h+1)(x+2h-1)\Delta_h^3 T_k(x) \\ &\qquad + [2(x+h+1)(x+h-1) - k^2h^2]\Delta_h^2 T_k(x) \\ &\qquad + (2x+3h-3k^2h)h\Delta_h T_k(x) - 2k^2h^2T_k(x) = 0, \end{aligned} \tag{3-12}$$

for $k=2m$. This equation holds for $k=2m+1$ as well. The proof in that case utilizes the first of equations (3-11) and (2-5), (2-6), (3-1), (3-3).

4. The minimax problem. We are now ready to establish the minimax property of the polynomials $T_k^{(h)}(x)$.

THEOREM 4. *Let Σ_h be the set of all subsets of the interval $[-1, +1]$ which contain the endpoints and at least one point in each closed subinterval of length h. Let Q_k be the set of all monic polynomials of degree $k>1$ with real coefficients. If*

$$0 < h < 2/(k-1), \tag{4-1}$$

then there is a set σ^* in Σ_h such that, for every polynomial q in Q_k and every set σ in Σ_h,

$$\begin{aligned} \max_{x\in\sigma} |q(x)| &\geqq \max_{x\in\sigma^*} |T_k^{(h)}(x)| \\ &= C_k(h) = 2^{1-k}\prod_{i=1}^{k}\left\{1+\left(i-\frac{k+1}{2}\right)h\right\}. \end{aligned} \tag{4-2}$$

PROOF. We note that the expression for $C_k(h)$ in (4-2) reduces to

(2-13a) when k is odd and to (2-14a) when k is even. In order to construct a set σ^* such that $\max_{x\in\sigma^*} |T_k^{(h)}(x)| = C_k$, we form the set σ_0 consisting of -1, $+1$, and all points nh for integral n such that $-1 < nh < 1$. Clearly, σ_0 is in Σ_h. From σ_0 we construct σ^* for k even by replacing each point nh falling in an interval (u_i, u_i+h) or (v_i, v_i+h) (notation of Theorem 2-2) by both endpoints of the interval. No ambiguity can arise as to which interval a given point lies in, since by (2-20), (2-21) no two of these intervals overlap. For k odd, σ^* is obtained from σ_0 by replacing points nh falling in intervals (r_i, r_i+h) or (s_i, s_i+h) by both endpoints of the interval. In this case, absence of overlap between intervals is assured by (2-17), (2-18). Since these replacements can only reduce the distance between consecutive points, σ^* also belongs to Σ_h. It follows from (4-1) and Theorem 3-1 or 3-2 that $|T_k(x)| \leqq C_k$ for all x in σ^* with equality holding for $x = \pm 1$ (and possibly other points of σ^*). Hence, $\max_{x\in\sigma^*} |T_k(x)| = C_k$.

Next, consider any set σ in Σ_h. By (2-16) and (2-19) there is at least one point of σ in each of the intervals, $[r_i, r_i+h]$, $[s_i, s_i+h]$ $[u_i, u_i+h]$, $[v_i, v_i+h]$. We select $k+1$ points from σ as follows: $x_0 = -1$, $x_k = +1$; for $k = 2m$, x_{2i} is any point of σ in $[u_i, u_i+h]$ $(i = 1, 2, \cdots, m-1)$, and x_{2i-1} is any point of σ in $[v_i, v_i+h]$ $(i = 1, 2, \cdots, m)$; for $k = 2m+1$, x_{2i} is any point of σ in $[s_i, s_i+h]$, and x_{2i-1} is any point of σ in $[r_i, r_i+h]$ $(i = 1, 2, \cdots, m)$. By Theorem 2-1 or 2-2,

$$-1 = x_0 < x_1 < \cdots < x_k = +1. \tag{4-3}$$

By (4-1) and Theorem 3-1 or 3-2,

$$\begin{aligned} T_k(x_j) &\geqq C_k \quad &&\text{for } j \equiv k \pmod 2, \\ T_k(x_j) &\leqq -C_k \quad &&\text{for } j \not\equiv k \pmod 2. \end{aligned} \tag{4-4}$$

Now let $q(x)$ be any polynomial in Q_k, and suppose that

$$|q(x_j)| < C_k, \qquad (j = 0, 1, \cdots, k). \tag{4-5}$$

Then by (4-4), $T_k(x_j) - q(x_j)$ would have the sign of $T_k(x_j)$—namely, positive for $j \equiv k \pmod 2$ and negative for $j \not\equiv k \pmod 2$. Hence by (4-3), $T_k(x) - q(x)$ changes sign k times in the interval $[-1, +1]$. But this is impossible, since $T_k - q$ is a polynomial of degree less than k. Consequently, (4-5) is false; so we have

$$\max_{x\in\sigma} |q(x)| \geqq \max_{j=0,1,\dots,k} |q(x_j)| \geqq C_k,$$

and the proof is complete.

As pointed out by the referee, the proof may be slightly modified to show that strict inequality holds in (4-2) unless q is $T_k^{(h)}$. Instead of (4-5) we suppose that

$$|q(x)| \leqq C_k, \quad (j = 0, 1, \cdots, k).$$

Our conclusion then is that $T_k(x) - q(x)$ has at least k zeros (not necessarily distinct) in $[-1, +1]$ and hence vanishes identically. On the other hand, σ^* is not unique. For σ^* in Σ_h, $\max_{x\in\sigma^*} |T_k(x)| = C_k$ if and only if σ^* contains both endpoints but no other points of each interval $[u_i, u_i+h]$ and $[v_i, v_i+h]$ for k even, $[r_i, r_i+h]$ and $[s_i, s_i+h]$ for k odd.

Corollary 4. *Let $c_1 < c_2 < \cdots < c_n$ be real numbers, let $L = c_n - c_1$ and $d = \max(c_{i+1} - c_i)$, and let $q(x)$ be any polynomial in Q_k. If $L > d(k-1)$,*

$$\max_{i=1,2,\ldots,n} |q(c_i)| \geqq 2^{1-2k} \prod_{i=1}^{k} \{L + d(2i - 1 - k)\}.$$

Proof. Let $y_i = (2c_i - c_1 - c_n)/L$, and define the polynomial $p(y) = (2/L)^k q(c_1 + (L/2)(y+1))$. Then, $\{y_1, y_2, \cdots, y_n\}$ belongs to $\sum_{2d/L}$, and $p(y)$ belongs to Q_k. Since $q(c_i) = (L/2)^k p(y_i)$, Theorem 4-1 yields

$$\max_{i=1,2,\ldots,n} |q(c_i)| \geqq (L/2)^k C_k(2d/L).$$

References

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