INEQUALITIES FOR THE DIFFERENCE BODY OF A CONVEX BODY

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1. **Introduction.** Let K be a convex body, i.e. a compact convex set having interior points, in n-dimensional Euclidean space E_n. The "difference body" of K, denoted by DK, is the centrally symmetric convex body (centered at the origin) defined by,

$$DK = K + (-K) = \{x - y: x \in K, y \in K\}. \tag{1}$$

It is well known that DK can equivalently be described as follows,

$$DK = \{x: (x + K) \cap K \neq \varnothing\}. \tag{2}$$

In the following, S will denote the boundary of the unit ball in E_n, and u a variable point of S (so u is a unit vector, or "direction"). The polar equation of the boundary of DK is given by $\rho = \rho(u)$, $u \in S$, so $\rho(u)$ is the radius of DK in the direction u. Then $\rho(u)$ is the maximum length of a chord of K having direction u—the length of a "diameter" of K having direction u.

Let μ denote n-dimensional Lebesque measure in E_n. Then Rogers and Shephard [5] proved that

$$\mu(DK) \leqq \binom{2n}{n} \mu(K), \tag{3}$$

where equality holds if and only if K is a simplex. They later gave a more geometrical proof in [6]. In §2 we give a proof of (3) which clarifies its relationship with the Brunn-Minkowski Theorem and leads to the following generalization, proved at the end of §3. Let $\alpha \geqq 1 - n$, and let $B(p, q)$ be the Beta function, defined by

$$B(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1} dt. \tag{4}$$

Then

$$\int_{K \times K} |x - y|^\alpha \, d\mu(x) d\mu(y) \geqq B(n + \alpha, n + 1) \mu(K) \int_S \rho^{n+\alpha}(u) \, du, \tag{5}$$

where $|x - y|$ is the distance, in E_n, between x and y. Equality holds in (5) if and only if K is a simplex. One sees that (3) follows from (5) by setting $\alpha = 0$.

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2. The proof of (3) given in [5] depended on a lemma of T. Bang [1, Lemma I]. One can derive that lemma from the following, proved by Fáry and Rédei [4].

LEMMA 1. *Let K_1 and K_2 be convex bodies in E_n. Let $\mathfrak{F}$ be the family of convex sets $(x+K_1)\cap K_2$, with x ranging over $B=\{x: (x+K_1)\cap K_2\neq\emptyset\}$. Then $\mathfrak{F}$ is a concave family.*

The Brunn-Minkowski Theorem then implies the following lemma, which is simply a restatement of [4, Satz 3].

LEMMA 2. *Let K_1 and K_2 be convex bodies in E_n, and let f be the function defined by $f(x)=\mu((x+K_1)\cap K_2)^{1/n}$, with domain $B=\{x: (x+K_1)\cap K_2\neq\emptyset\}$. Then f is a concave function on B.*

The following proof of (3) is only a slight variant of the first proof of Rogers and Shephard.

PROOF OF (3). Let (r, u) be spherical coordinates for E_n, so r represents distance from 0 and u ranges over the unit sphere S centered at 0. Let $f(x)=\mu((x+K)\cap K)^{1/n}$, for $x\in DK$. Then f is concave, by Lemma 2, and $(f(0))^n=\mu(K)$. Let $g(x)=f(0)(1-r/\rho)$, $x\in DK$, where $r=|x|$ is the distance from x to 0, and $\rho=\rho(x/r)$ is the radius of DK in the direction of x. Then g vanishes on the boundary of DK, $g(0)=f(0)$, and g is linear on each segment joining 0 to the boundary of DK (the ordinate set of g is a cone in E_{n+1} with altitude $f(0)$, base DK, and is contained in the ordinate set of f). The concavity of f implies $f(x)\geqq g(x)$, $x\in DK$. Thus

$$
\begin{aligned}
\int_{DK}\mu((x+K)\cap K)d\mu(x) &= \int_{DK} f^n d\mu \geqq \int_{DK} g^n d\mu \\
&= (f(0))^n\int_S\int_0^\rho (1-r/\rho)^n r^{n-1}drdu \\
&= \mu(K)\int_S \rho^n\int_0^1 t^{n-1}(1-t)^n dtdu \\
&= \binom{2n}{n}^{-1}\mu(K)\,\frac{1}{n}\int_S \rho^n du = \binom{2n}{n}^{-1}\mu(K)\mu(DK).
\end{aligned}
\tag{6}
$$

With the following easy identity (see [5, (7)]),

$$
\mu(K)^2=\int_{DK}\mu((x+K)\cap K)d\mu(x),
\tag{7}
$$

the inequality (3) follows. Equality holds in (3) if and only if equality holds throughout in (6), so that $f=g$. Then f is linear on each segment joining 0 to the boundary of DK. By the Brunn-Minkowski Theorem, this happens if and only if $(x+K)\cap K$ is positively homothetic to $K=(0+K)\cap K$ whenever $(x+K)\cap K$ is nondegenerate. It is proved in [**5**] that this property characterizes simplices; hence equality holds in (3) if and only if K is a simplex. (The proof of this characterization of simplices is simplified in [**3**].) This completes the proof.

The idea used in the above proof suggests the following theorem, whose proof is omitted since it follows precisely the same lines as above.

THEOREM 1. *Let f be a nonnegative concave function defined on a convex body B in E_n. Let $x_0\in B$, and let $h(\xi)$, $\xi\geqq 0$, be a strictly increasing real-valued function of a real variable. Then*

$$\int_B h(f(x))d\mu(x) \geqq n\mu(B)\int_0^1 h(tf(x_0))(1-t)^{n-1}dt, \tag{8}$$

where equality holds if and only if f vanishes on the boundary of B and is linear on each segment joining x_0 to the boundary of B.

Note that (3) follows from (8) by taking $B=DK$, $x_0=0$, $f(x)=\mu((x+K)\cap K)^{1/n}$, $h(\xi)=\xi^n$.

The following theorem, which was used in [**6**] in order to prove (3), follows immediately from Theorem 1.

THEOREM 2. *Let K be an $(r+s)$-dimensional convex body. Let R be an r-dimensional section of K with r-measure $\mu(R)$, and let S be the projection of K onto an s-dimensional subspace totally orthogonal to R. Let $\mu(S)$ be the s-measure of S. Then*

$$\binom{r+s}{s}\mu(K) \geqq \mu(R)\mu(S), \tag{9}$$

where equality holds if and only if each r-dimensional section of K parallel to R is positively homothetic to R.

PROOF. In Theorem 1, take $n=s$, $B=S$, $h(\xi)=\xi^r$, $f(x)=\mu(R(x))^{1/r}$, for $x\in S$, where $R(x)$ is the r-dimensional section of K by the r-flat orthogonal to S passing through x, and $\mu(R(x))$ is the r-measure of $R(x)$. Let $R(x_0)=R$. By the Brunn-Minkowski Theorem, f is concave on S. Hence, from (8), we have,

$$\begin{aligned}\mu(K) &= \int_S \mu(R(x))d\mu(x) \geqq s\mu(S)\mu(R)\int_0^1 t^r(1-t)^{s-1}dt \\ &= \mu(S)\mu(R)\binom{r+s}{s}^{-1}.\end{aligned} \tag{10}$$

Equality holds if and only if f is linear on each segment joining x_0 to the boundary of S. This is the case if and only if all the $R(x)$ are homothetic. This completes the proof.

3. The position of an oriented line G in E_n can be fixed by specifying its direction u and the point p where G intersects the orthogonal hyperplane through the origin. The "integral-geometric density" dG, for oriented lines is then given by

$$dG = dp\,du, \tag{11}$$

where dp is the $(n-1)$-dimensional volume element in the orthogonal hyperplane, and du is the element of surface area of the unit sphere S. If K is a convex body in E_n, let $\sigma(G)=\sigma(p, u)$ be the length of the chord $K\cap G$, for each oriented line $G=G(p, u)$, where p and u are as described above. Let

$$I_\alpha(K) = \int \sigma^\alpha(G)dG = \int_S \int_{K_u} \sigma^\alpha dp\,du, \tag{12}$$

where K_u is the orthogonal projection of K onto a hyperplane orthogonal to u, and dp is the volume element in that hyperplane.

Theorem 3. *Let K be a convex body in E_n, and let $\rho=\rho(u)$, $u\in S$, be the polar equation of DK. Then if m is a real number $\geqq 1$, we have*

$$I_{m+1}(K) \geqq m(m+1)B(m, n+1)\mu(K) \int_S \rho^m(u)du, \tag{13}$$

where equality holds if and only if K is a simplex.

Proof. The concavity of $\mu((x+K)\cap K)^{1/n}$ implies (see the proof of (3) in §2)

$$\mu((x+K)\cap K) \geqq \mu(K)(1 - r/\rho)^n, \tag{14}$$

with $r=|x|$, and $\rho=\rho(x/r)=\rho(u)$, $x\in DK$. Multiply both sides of (14) by r^{m-1} and integrate with respect to r, to obtain

$$\begin{aligned} \int_0^\rho r^{m-1}\mu((x+K)\cap K)dr &\geqq \mu(K)\int_0^\rho r^{m-1}(1 - r/\rho)^n dr \\ &= \mu(K)B(m, n+1)\rho^m(u). \end{aligned} \tag{15}$$

Let $l(p, u)$ be the length of the intersection of the line $G(p, u)$ with $(x+K)\cap K$, where $u=x/|x|$. Then $l(p, u)=\sigma(p, u)-r$, when $\sigma\geqq r$, and

$$
\begin{aligned}
\int_0^\rho r^{m-1}\mu((x+K)\cap K)dr &= \int_0^\rho r^{m-1}\int_{K_u} l(p,u)\,dp\,dr \\
&= \int_{K_u}\int_0^\rho r^{m-1} l(p,u)\,dr\,dp \\
&= \int_{K_u}\int_0^\sigma r^{m-1}(\sigma - r)\,dr\,dp \\
&= \frac{1}{m(m+1)}\int_{K_u}\sigma^{m+1}(p,u)\,dp.
\end{aligned}
\tag{16}
$$

Then (15) and (16) yield

$$
\frac{1}{m(m+1)}\int_{K_u}\sigma^{m+1}(p,u)\,dp \geqq \mu(K)B(m, n+1)\rho^m(u). \tag{17}
$$

Integration of (17) with respect to u yields (13). Equality holds in (13) if and only if it holds in (14), for all $x \in DK$. It follows (as in the proof of (3) in §2) that this happens if and only if K is a simplex. This completes the proof.

In (16), set $m=n$ and integrate both sides over S. This gives

$$
\int_S\int_0^\rho \mu((x+K)\cap K)r^{n-1}\,dr\,du = \frac{1}{n(n+1)} I_{n+1}(K). \tag{18}
$$

Using (7), we then have

$$
I_{n+1}(K) = n(n+1)\mu(K)^2. \tag{19}
$$

This is a generalization to E_n of an integral-geometric formula of Crofton (see [2, p. 20]). If we set $m=n$ in (13) and use (19), we obtain the Rogers-Shephard inequality (3).

In order to prove (5), we need a generalization of the formula of Crofton given in [2, p. 19]. The integral-geometric density for pairs of points $x, y \in E_n$ is $d\mu(x)d\mu(y)$. If $G = G(p, u)$ is the line through x and y, then a straightforward computation shows that

$$
d\mu(x)d\mu(y) = |x-y|^{n-1}dt_1dt_2\,dp\,du, \tag{20}
$$

where $|x-y|$ is the distance from x to y, t_1 is the distance of x from a fixed point of G, and t_2 is the distance of y from a fixed point of G. Multiplying both sides of (20) by $|x-y|^\alpha$, $\alpha \geqq 1-n$, and integrating over all pairs of points $x, y \in K$ yields

$$(21)\quad \int_{K\times K} |x-y|^\alpha d\mu(x)d\mu(y) = \frac{1}{(n+\alpha)(n+\alpha+1)} I_{n+\alpha+1}(K).$$

If we set $m=n+\alpha$ in (13), and apply (21), we obtain (5).

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