A GENERALIZATION OF THE GÖLLNITZ-GORDON PARTITION THEOREMS

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1. Introduction. Among the most striking results in the theory of partitions are the Rogers-Ramanujan identities [9, p. 291]. These may be stated combinatorially as follows.

(1.1) The number of partitions of \( n \) with minimal difference 2 is equal to the number of partitions of \( n \) into parts of the forms \( 5m+1 \) and \( 5m+4 \).

(1.2) The number of partitions of \( n \) into parts not less than 2, and with minimal difference 2, is equal to the number of partitions of \( n \) into parts of the forms \( 5m+2 \) and \( 5m+3 \).

In 1926, I. J. Schur proved the following theorem which is similar to the above results [10].

(1.3) The number of partitions of \( n \) of the form \( n = b_1 + \cdots + b_s \), where \( b_i - b_{i+1} \geq 3 \), and \( b_i - b_{i+1} > 3 \) if \( 3 \mid b_i \), is equal to the number of partitions of \( n \) into parts of the forms \( 6m+1 \) and \( 6m+5 \).

However, in 1948 H. L. Alder shut the door on further generalizations in this direction by proving the following three theorems [1]. Here \( g_{d,m}(n) \) is the number of partitions of \( n \) into parts differing by at least \( d \), each part being greater than or equal to \( m \).

(1.4) Let \( S \) be any fixed set of positive integers, then \( g_{d,m}(n) \) is not always equal to the number of partitions of \( n \) into parts taken from \( S \) if \( d > 2 \).

(1.5) Let \( S \) be any fixed set of positive integers, then \( g_{d,m}(n) \) is not equal to the number of partitions of \( n \) into distinct parts taken from \( S \) if \( d > 1 \).

(1.6) Let \( S \) be any fixed set of positive integers, then the number of partitions of \( n \) into parts differing by at least \( d \) and where no consecutive multiples of \( d \) appear is not equal to the number of partitions of \( n \) into parts taken from \( S \) if \( d > 3 \).

The case of (1.6) in which \( d = 2 \) was treated independently by H. Göllnitz [6, p. 33–34] in 1960 and by B. Gordon [8, p. 741] in 1965. They proved the following two identities.

(1.7) The number of partitions of any positive integer \( n \) into parts \(-1, 4, 7 \mod 8\) is equal to the number of partitions of the form \( n = b_1 + \cdots + b_s \), where \( b_i - b_{i+1} \geq 2 \), and \( b_i - b_{i+1} > 2 \) if \( b_i \) is even.

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(1.8) The number of partitions of any positive integer $n$ into parts $\equiv 3, 4, 5 \pmod{8}$ is equal to the number of partitions of the form $n = b_1 + \cdots + b_s$ satisfying $b_s \geq 3$ in addition to the inequalities of (1.7).

A different form of generalization of the Rogers-Ramanujan identities was discovered in 1961 by B. Gordon [7]. He proved the following result.

(1.9) Let $a$ and $k$ be integers with $0 < a \leq k$. Let $A_{k,a}(n)$ denote the number of partitions of $n$ into parts not of the forms $(2k + 1)m$, $(2k + 1)m \pm a$. Let $B_{k,a}(n)$ denote the number of partitions of $n$ of the form $n = \sum_{i=1}^{\infty} f_i \cdot i$ (here $f_i$ is the number of times the part $i$ appears in the partition) with $f_1 \leq a - 1$ and for all $i \geq 1$,

$$f_i + f_{i+1} \leq k - 1.$$ 

Then $A_{k,a}(n) = B_{k,a}(n)$.

When $k = a = 2$, (1.9) reduces to (1.1), and when $k = 2, a = 1$, (1.9) reduces to (1.2). Further theorems of this nature have been proved in subsequent papers [2], [3], [4].

The object of this paper is to generalize the Göllnitz-Gordon identities, (1.7) and (1.8), in the same manner that (1.9) generalizes (1.1) and (1.2). Our main result is the following theorem.

**Theorem 1.** Let $a$ and $k$ be integers with $0 < a \leq k$. Let $C_{k,a}(n)$ be the number of partitions of $n$ into parts which are neither $\equiv 2 \pmod{4}$ nor $\equiv 0, \pm (2a - 1) \pmod{4k}$. Let $D_{k,a}(n)$ denote the number of partitions of $n$ of the form $n = \sum_{i=1}^{\infty} f_i \cdot i$ with $f_1 + f_2 \leq a - 1$ and for all $i \geq 1$,

$$f_{2i-1} \leq 1 \quad \text{and} \quad f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1.$$ 

Then $C_{k,a}(n) = D_{k,a}(n)$.

When $k = a = 2$, the theorem reduces to (1.7), and when $k = 2, a = 1$, the theorem reduces to (1.8). As an example, if $k = a = 3$, the seven partitions enumerated by $D_{3,3}(8)$ are $8$, $7 + 1$, $6 + 2$, $5 + 3$, $5 + 2 + 1$, $4 + 4$, $4 + 3 + 1$; the seven partitions enumerated by $C_{3,3}(8)$ are $8$, $4 + 4$, $4 + 3 + 1$, $4 + 1 + 1 + 1 + 1$, $3 + 3 + 1 + 1$, $3 + 1 + 1 + 1 + 1 + 1$, $1 + 1 + 1 + 1 + 1 + 1$.

In §2, we shall prove Theorem 1. In §3, we shall prove some analogues of the analytic form of the Rogers-Ramanujan identities [9, p. 290].

2. **Proof of Theorem 1.** We shall study the following functions. Throughout, $|q| < 1$, and $x \neq -q^{2n+1}$ for any $n \geq 1$. 

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\[ E_{k,i}(x) = \sum_{n=0}^{\infty} (-1)^n x^n q^{2kn^2 + (2k-2i+1)n} \left( \prod_{j=0}^{n-1} \frac{1 + q^{2j+1}}{1 - xq^{2j+2}} \right) \]

\[ F_{k,i}(x) = \sum_{n=0}^{\infty} (-1)^n x^n q^{2kn^2 + (1-2i)n} \frac{(1 - xq^{4in})}{(1 - xq^{2n})} \]

\[ H_{k,i}(x) = F_{k,i}(x) \prod_{j=1}^{\infty} \frac{1 + xq^{2j-1}}{1 - xq^{2j}} \]

\[ J_{k,i}(x) = E_{k,i}(x) \prod_{j=1}^{\infty} \frac{1 + xq^{2j-1}}{1 - xq^{2j}} \]

From the above definitions, we have immediately

\[ F_{k,0}(x) = H_{k,0}(x) = 0, \]

and

\[ H_{k,i}(0) = J_{k,i}(0) = 1, \quad 1 \leq i \leq k. \]

To prove Theorem 1 we shall need the following lemmas.

**Lemma 1.** \( H_{k,i}(x) - H_{k,i-1}(x) = x^{i-1}J_{k,k-i+1}(x). \)

**Proof.** We prove equivalently

\[ F_{k,i}(x) - F_{k,i-1}(x) = x^{i-1}E_{k,k-i+1}(x). \]

\[ F_{k,i}(x) - F_{k,i-1}(x) = \sum_{n=0}^{\infty} (-1)^n x^n q^{2kn^2 + n} \left( \prod_{j=0}^{n-1} \frac{1 + q^{2j+1}}{1 - xq^{2j+2}} \right) \]

\[ \cdot \left( q^{-2in} - x^i q^{2in} - q^{-2in+2n} + x^{-1} q^{2in-2n} \right) \]

\[ = \sum_{n=0}^{\infty} (-1)^n x^n q^{2kn^2 + n} \left( \prod_{j=0}^{n-1} \frac{1 + q^{2j+1}}{1 - xq^{2j+2}} \right) \]

\[ \cdot \left[ \{ q^{-2in}(1 - q^{2n}) \} + \{ x^{-1} q^{2in-2n}(1 - xq^{2n}) \} \right]. \]

We now split our sum into two separate parts and replace \( n \) by \( n + 1 \) in the first part. Hence
Thus we have Lemma 1.

**Lemma 2.**

\[ J_{k,i}(x) = H_{k,i}(xq^2) + xqH_{k,i-1}(xq^2). \]

**Proof.** We prove equivalently

\[ E_{k,i}(x) = (1 - xq^2)(1 + xq)^{-1}(F_{k,i}(xq^2) + xqF_{k,i-1}(xq^2)). \]

\[ E_{k,i}(x) = \sum_{n=0}^{\infty} (-1)^n x^k q^{2kn^2 + (2k-2i+1)n} \left( \prod_{j=0}^{n-1} \frac{(1 + q^{2j+1})(1 - xq^{2j+2})}{(1 + xq^{2j+1})(1 - q^{2j+2})} \right) \cdot \Bigl(1 - xq^{2n+1} + xq^{2n+1}(1 - x^{i-1}q^{2n+1}2(i-1)) \Bigr) \cdot \frac{1}{1 + xq^{2n+1}} \]

\[ = (1 - xq^2)(1 + xq)^{-1}(F_{k,i}(xq^2) + xqF_{k,i-1}(xq^2)). \]
Thus we have Lemma 2.

**Lemma 3.** \( J_{k,1}(x) = J_{k,k}(xq^2) \).

**Proof.** In Lemma 1, put \( i = 1 \); then (since \( H_{k,0}(x) = 0 \) by (2.5)) \( H_{k,1}(x) = J_{k,k}(x) \). In Lemma 2, put \( i = 1 \); thus \( J_{k,1}(x) = H_{k,1}(xq^2) \). Combining these two results, we obtain Lemma 3.

**Lemma 4.** \( J_{k,i}(x) - J_{k,i-1}(x) = x^{i-1}q^{2i-2}(qJ_{k,k-i+1}(xq^2) + J_{k,k-i+2}(xq^2)) \).

**Proof.** By Lemma 2,

\[
J_{k,i}(x) - J_{k,i-1}(x) = (H_{k,i}(xq^2) - H_{k,i-1}(xq^2)) + xq(H_{k,i-1}(xq^2) - H_{k,i-2}(xq^2)) = x^{i-1}q^{2i-2}(qJ_{k,k-i+1}(xq^2) + J_{k,k-i+2}(xq^2)),
\]

where the second equation follows from Lemma 1. Thus we have Lemma 4.

We are now ready to treat our main theorem.

**Proof of Theorem 1.** We may expand \( J_{k,i}(x) \) as follows

\[
J_{k,i}(x) = \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} c_{k,i}(M, N)x^M q^N,
\]

where the double sum is subject to those conditions listed before (2.1).

Then by means of Lemmas 3 and 4 and equations (2.1) and (2.4), we easily verify that for \( 1 \leq i \leq k \)

\[
c_{k,i}(M, N) = \begin{cases} 1 & \text{if } M = N = 0, \\ 0 & \text{if either } M \leq 0 \text{ or } N \leq 0, \text{ and } M^2 + N^2 \neq 0, \\ c_{k,1}(M, N) = c_{k,k}(M, N - 2M), \\ c_{k,i}(M, N) - c_{k,i-1}(M, N) \\ = c_{k,k-i+1}(M - i + 1, N - 2M) \\ + c_{k,k-i+2}(M - i + 1, N - 2M + 1), & 1 < i \leq k. \end{cases}
\]

One easily verifies by mathematical induction that the \( c_{k,i}(M, N) \) for \( 1 \leq i \leq k \) are uniquely determined by (2.8), (2.9), and (2.10).

Let \( p_k(a, M, N) \) denote the number of partitions of \( N \) into \( M \) parts of the form \( N = \sum_{i=1}^{\infty} f_i \cdot i \) with \( f_1 + f_2 \leq a - 1 \) and for all \( i \geq 1, f_{2i-1} \leq 1 \) and \( f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1 \). We wish to show that the \( p_{k,i}(M,N) \) satisfy (2.8), (2.9), and (2.10). Now (2.8) is satisfied by definition.

As for (2.9), let us consider any partition enumerated by \( p_{k,1}(M, N) \). Since neither 1 nor 2 appears, every summand is \( \geq 3 \). Subtracting 2 from every summand, we obtain a partition of \( N - 2M \)
into $M$ parts with $f_1 + f_2 \leq k - 1$ and $f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1$, $f_{2i-1} \leq 1$. Thus we have a partition of the type enumerated by $p_{k,k}(M, N - 2M)$. The above procedure establishes a one-to-one correspondence between the partitions enumerated by $p_{k,1}(M, N)$ and the partitions enumerated by $p_{k,k}(M, N - 2M)$. Thus

\[ p_{k,1}(M, N) = p_{k,k}(M, N - 2M). \]

Finally we treat (2.10). We note that $p_{k,a}(M, N) - p_{k,a-1}(M, N)$ enumerates the number of partitions of $N$ into $M$ parts of the form $N = \sum_{i=1}^{\infty} f_i \cdot i$ with $f_1 + f_2 = a - 1$ and $f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1$, $f_{2i-1} \leq 1$. Hence either $f_2 = a - 1$, or $f_1 = 1$ and $f_2 = a - 2$. In case $f_2 = a - 1$, we see that $f_3 + f_4 \leq k - 1 - (a - 1)$; subtracting 2 from every summand, we obtain a partition of $N - 2M$ into $M - a + 1$ parts with $f_1 + f_2 \leq (k - a + 1) - 1$ and for all $i \geq 1$, $f_{2i-1} \leq 1$ and $f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1$. Thus we have a partition of the type enumerated by $p_{k,k-a+1}(M - a + 1, N - 2M)$. In case $f_2 = a - 2$ and $f_1 = 1$, we see that $f_3 + f_4 \leq k - 1 - (a - 2)$; subtracting 2 from every summand $\geq 2$ and removing the summand 1, we obtain a partition of $(N - 2M)$ into $M - a + 1$ parts with $f_1 + f_2 \leq (k - a + 2) - 1$ and for all $i \geq 1$, $f_{2i-1} \leq 1$ and $f_{2i} + f_{2i+1} + f_{2i+2} \leq k - 1$. Thus we have a partition of the type enumerated by $p_{k,k-a+2}(M - a + 1, N - 2M + 1)$. The above procedure establishes a one-to-one correspondence between the partitions enumerated by $p_{k,a}(M, N) - p_{k,a-1}(M, N)$ and the partitions enumerated by $p_{k,k-a+1}(M - a + 1, N - 2M) + p_{k,k-a+2}(M - a + 1, N - 2M + 1)$. Hence

\[ p_{k,a}(M, N) - p_{k,a-1}(M, N) = p_{k,k-a+1}(M - a + 1, N - 2M) + p_{k,k-a+2}(M - a + 1, N - 2M + 1). \]

Thus by the comment following (2.10),

\[ c_{k,i}(M, N) = p_{k,i}(M, N), \quad 1 \leq i \leq k. \]

Thus for $1 \leq a \leq k$

\[
1 + \sum_{N=1}^{\infty} C_{k,a}(N)q^N = \prod_{n=1; n \neq 2 \pmod{4}; n \neq 0, \pm (2a-1) \pmod{4k}} (1 - q^n)^{-1} = J_{k,a}(1) = \sum_{N=-\infty}^{\infty} \sum_{M=-\infty}^{\infty} p_{k,a}(M, N)q^N = 1 + \sum_{N=1}^{\infty} D_{k,a}(N)q^N,
\]
where the second equation follows from Jacobi’s identity [9, p. 283]. Therefore $C_{k,a}(N) = D_{k,a}(N)$. This concludes the proof of Theorem 1.

3. Analytic identities. Since $J_{1,1}(x) = J_{1,1}(xq^2)$ and $\lim_{a \to 0} J_{1,1}(a) = 1$, we have

$$F_{1,1}(x) = \sum_{n=0}^{\infty} (-1)^n x^n q^{2n^2-n} (1 - xq^{4n}) (1 - xq^{2n})^{-1}$$

$$\prod_{j=0}^{n-1} \frac{(1 + q^{2j+1})(1 - xq^{2j+2})}{(1 + xq^{2j+1})(1 - q^{2j+2})}$$

$$E_{1,1}(x) = \prod_{n=1}^{\infty} \frac{(1 - xq^{2n})}{(1 + xq^{2n-1})}.$$  

(3.1)

It is easily deduced from Lemmas 3 and 4 that

$$J_{2,2}(x) = (1 + xq)J_{2,2}(xq^2) + xq^2J_{2,2}(xq^4).$$  

(3.2)

Expanding $J_{2,2}(x)$ in powers of $x$ and using (3.2) and $J_{2,2}(0) = 1$, we obtain

$$J_{2,2}(x) = \sum_{n=0}^{\infty} \frac{x^n q^{2n^2} (1 + q)(1 + q^3) \cdots (1 + q^{2m-1})}{(1 - q^2)(1 - q^4) \cdots (1 - q^{2m})}$$

$$= H_{2,1}(x) = \prod_{m=1}^{n} \frac{(1 + xq^{2m-1})}{(1 - xq^{2m})} \sum_{n=0}^{\infty} (-1)^n x^n q^{4n^2-n}$$

$$\cdot \prod_{j=0}^{n-1} \frac{(1 + q^{2j+1})(1 - xq^{2j+2})}{(1 + xq^{2j+1})(1 - q^{2j+2})}.$$  

(3.3)

(3.3) was the identity from which Göllnitz originally deduced (1.7) and (1.8).

When $k = 3$, the related $q$-identity becomes more complicated. If in equation (10.1) of [5, p. 431] we replace $x$ by $q$ and then put $a = x$, $f = -x$, we obtain

$$E_{3,3}(x) = \prod_{m=1}^{\infty} (1 - xq^m) \sum_{n=0}^{\infty} x^n q^{2n^2} \prod_{j=0}^{n-1} \frac{(1 + xq^{2j})}{(1 - q^{2j+1})(1 - xq^{2j+1})(1 + xq)}$$

$$= F_{3,1}(x) = \sum_{n=0}^{\infty} (-1)^n x^n q^{6n^2-n} \frac{(1 - xq^{4n})}{(1 - xq^{2n})}$$

$$\cdot \prod_{j=0}^{n-1} \frac{(1 + q^{2j+1})(1 - xq^{2j+2})}{(1 + xq^{2j+1})(1 - q^{2j+2})}.$$  

(3.4)

Putting $x = 1$ in (3.1), we obtain a special case of Jacobi’s identity [9, p. 283]. Putting $x = 1$ in (3.3), we obtain equation (36) of [11, p.
Putting \( x = g^2 \) in (3.3), we obtain equation (34) of [11, p. 155].
Putting \( x = g^2 \) in (3.4), we obtain equation (49) of [11, p. 156].
Putting \( x = 1 \) in (3.4), we obtain equation (54) of [11, p. 157] (there appears to be a minor misprint in Slater's equation (54) which is easily corrected).

References


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