AN INEQUALITY FOR LINEAR TRANSFORMATIONS

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1. Statements of results. In this paper the following elementary inequality is proved and exploited.

**Theorem 1.** If $L$ is a positive-definite hermitian transformation on the finite dimensional unitary space $V$ and $p \geq 1$, then for arbitrary vectors $u$ and $v$

\[
\|u\|^2 + (L^{-p}v, v) \geq ((I + L)^{-p}u + v, u + v).
\]

From (1) we can conclude

**Theorem 2.** If $H$ and $K$ are positive-definite hermitian transformations on $V$ and $x$ and $y$ are arbitrary vectors, then

\[
(H^{-1}x, x) + (K^{-1}y, y) \geq ((H + K)^{-1}x + y, x + y).
\]

By a trivial induction on (2) the following corollary is obtained.

**Corollary.** If $A_k$, $k = 1, \ldots, m$, are positive-definite hermitian on $V$ and $x_1, \ldots, x_m$ are arbitrary vectors in $V$, then

\[
\sum_{k=1}^{m} (A_k^{-1}x_k, x_k) \geq \left(\sum_{k=1}^{m} A_k\right)^{-1} \left(\sum_{k=1}^{m} x_k, \sum_{k=1}^{m} x_k\right).
\]

The result (2) implies the following extension of Bergstrom's inequality [1, p. 119].

**Theorem 3.** Let $A$ and $B$ be $n$-square hermitian matrices and let $A_1$ be the principal submatrix of $A$ obtained by deleting row one and column one of $A$. If $A_1$ and $B_1$ (defined similarly) are positive-definite hermitian, then

\[
\frac{\det(A + B)}{\det(A_1 + B_1)} \geq \frac{\det(A)}{\det(A_1)} + \frac{\det(B)}{\det(B_1)}.
\]

We also prove

**Theorem 4.** Let $H$ and $K$ be positive-definite hermitian transformations on $V$ with eigenvalues $h_1 \geq \cdots \geq h_n$, $k_1 \geq \cdots \geq k_n$ respectively. If $H + K$ has eigenvalues $\tau_1 \geq \cdots \geq \tau_n$ and $m \leq n/2$, then

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2. Proofs. In proving (1) we establish a substantially more general result.

**Theorem 5.** Let $L$ be a positive-definite hermitian transformation on the unitary space $V$, dim $V = n$. Let $f$ be a scalar valued function defined on $(0, \infty)$ satisfying

$$(6) \quad f(1) = 1, \quad f(x) > 0, \quad f(1 + x) \geq 1 + f(x).$$

Then for arbitrary vectors $u$ and $v$

$$(7) \quad \|u\|^2 + (f(L)^{-1}v, v) \geq (f(I + L)^{-1}u + v, u + v).$$

**Proof.** Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of $L$ with a corresponding orthonormal set of eigenvectors $e_1, \cdots, e_n$. Let $\alpha_t = (u, e_t)$, $\beta_t = (v, e_t)$, $t = 1, \cdots, n$, and compute that

$$\|u\|^2 + (f(L)^{-1}v, v) - (f(I + L)^{-1}u + v, u + v)$$

$$= \sum_{t=1}^{n} \left\{ \frac{1}{f(\lambda_t)} \left| \alpha_t \right|^2 + \frac{1}{f(1 + \lambda_t)} \left| \alpha_t + \beta_t \right|^2 \right\}$$

$$= \sum_{t=1}^{n} \left\{ f(\lambda_t) f(1 + \lambda_t) \left| \alpha_t \right|^2 + f(1 + \lambda_t) \left| \beta_t \right|^2 \right.$$ 

$$- f(\lambda_t) \left| \alpha_t + \beta_t \right|^2 \right\} / f(\lambda_t) f(1 + \lambda_t)$$

$$\geq \sum_{t=1}^{n} \left\{ f(\lambda_t) (1 + f(\lambda_t)) \left| \alpha_t \right|^2 + (1 + f(\lambda_t)) \left| \beta_t \right|^2 \right.$$ 

$$- f(\lambda_t) \left( \left| \alpha_t \right|^2 + \left| \beta_t \right|^2 + 2 \left| \alpha_t \right| \left| \beta_t \right| \right) \right\} / f(\lambda_t) f(1 + \lambda_t)$$

$$= \sum_{t=1}^{n} \left\{ (f(\lambda_t))^2 \left| \alpha_t \right|^2 + \left| \beta_t \right|^2 - 2 \left| \alpha_t \right| \left| \beta_t \right| f(\lambda_t) \right\} / f(\lambda_t) f(1 + \lambda_t)$$

$$= \sum_{t=1}^{n} \left\{ (f(\lambda_t) \left| \alpha_t \right| - \left| \beta_t \right|)^2 \right\} / f(\lambda_t) f(1 + \lambda_t)$$

$$\geq 0.$$
\[(H + K)^{-1}x + y, x + y\) = \((H + K)^{-1}H^{1/2}(u + v), H^{1/2}(u + v)\)
\[
= (H^{1/2}(H + K)^{-1}H^{1/2}u + v, u + v) = ((H^{-1/2}(H + K)H^{-1/2})^{-1}u + v, u + v)
\]
\[
= ((I + L)^{-1}u + v, u + v) = \left\| u \right\|^2 + (L^{-1}v, v)
\]
\[
= (H^{-1}x, x) + (H^{1/2}K^{-1}H^{1/2}v, v) = (H^{-1}x, x) + (K^{-1}y, y).
\]

To prove Theorem 3 we derive an elementary identity for the determinant of an \(n\)-square matrix. Thus let \(X\) be an \(n\)-square matrix and suppose \(X(1|1)\) denotes the \((n - 1)\)-square principal submatrix of \(X\) obtained by deleting row 1 and column 1 of \(X\). More generally, if \(i_1 < \cdots < i_r, j_1 < \cdots < j_s\), are integers between 1 and \(n\) let \(X(i_1, \cdots, i_r|j_1, \cdots, j_s)\) denote the submatrix of \(X\) obtained by deleting rows \(i_1, \cdots, i_r\) and columns \(j_1, \cdots, j_s\) of \(X\). Now

\[
\det(X) = \sum_{j=1}^{n} (-1)^{1+j}x_{1j} \det(X(1|j))
\]
\[
= x_{11} \det(X(1|1)) + \sum_{j=2}^{n} (-1)^{1+j}x_{1j} \det(X(1|j)).
\]

Now

\[
\det(X(1|j)) = \sum_{k=2}^{n} (-1)^{k}x_{k1} \det(X(1,k|1,j)).
\]

Let \(c_{jk} = (-1)^{j+k}x_{k1} \det(X(1,k|1,j)), k, j = 2, \cdots, n\), so that the \((n - 1)\)-square matrix \(C = (c_{jk})\) is the adjugate of \(X(1|1)\), i.e. \(C = \text{adj } X(1|1)\). Then substituting (9) in (8) produces

\[
\det(X) = x_{11} \det(X(1|1))
\]
\[
+ \sum_{j=2}^{n} x_{1j} (-1)^{1+j} \sum_{k=2}^{n} (-1)^{k}x_{k1} \det(X(1,k|1,j))
\]
\[
= x_{11} \det(X(1|1)) - \sum_{j,k=2}^{n} x_{1j}x_{k1}c_{jk}.
\]

Thus, if \((,\, )\) denotes the standard inner product in the space of \((n - 1)\)-tuples over the complex numbers, (10) reads

\[
\det(X) = x_{11} \det(X(1|1)) - (\text{adj } X(1|1)u, v)
\]
in which \(u = (x_{21}, x_{31}, \cdots, x_{n1})\); \(v = (\bar{x}_{12}, \bar{x}_{13}, \cdots, \bar{x}_{1n})\). In case \(X = A\) is hermitian we know that \(u = v\) and we have in the notation of Theorem 3 with \(U_A = (a_{21}, a_{31}, \cdots, a_{n1})\)

\[
\det(A) = a_{11} \det(A_1) - (\text{adj } A_{1}u_A, u_A).
\]
If we assume that $A_i$ (and $B_i$) are positive-definite hermitian, then of course $\text{adj } A_i$ is also, $\det(A_i) > 0$, $\text{adj } A_i = \det(A_i) A_i^{-1}$ and we have from (11) (applied to $A$, $B$ and $A + B$)

$$\frac{\det(A)}{\det(A_i)} = a_{11} - (A_i^{-1} u_A, u_A),$$

$$\frac{\det(B)}{\det(B_i)} = b_{11} - (B_i^{-1} u_B, u_B),$$

$$\frac{\det(A + B)}{\det(A_i + B_i)} = a_{11} + b_{11} - ((A_i + B_i)^{-1} u_{A+B}, u_{A+B}).$$

Now, $u_{A+B} = u_A + u_B$ so that

$$\frac{\det(A + B)}{\det(A_i + B_i)} - \frac{\det(A)}{\det(A_i)} - \frac{\det(B)}{\det(B_i)}$$

$$= (A_i^{-1} u_A, u_A) + (B_i^{-1} u_B, u_B) - ((A_i + B_i)^{-1} u_A + u_B, u_A + u_B)$$

and we may apply (2) to complete the proof of (4).

To prove Theorem 4 let $x_1, \ldots, x_m$ be an orthonormal set of eigenvectors of $H$ corresponding respectively to $h_1, \ldots, h_m$. Let $y_1, \ldots, y_m$ be an orthonormal set of vectors in the orthogonal complement of the space spanned by $x_1, \ldots, x_m$ (possible since $m \leq n/2$). Then using a result due to Fan [1, p. 114], we have from (2)

$$\sum_{j=1}^{m} \left( \frac{1}{h_j} + \frac{1}{h_{n-j+1}} \right) \geq \sum_{j=1}^{m} (H^{-1} x_j, x_j) + \sum_{j=1}^{m} (K^{-1} y_j, y_j)$$

$$\geq \sum_{j=1}^{m} ((H + K)^{-1} x_j + y_j, x_j + y_j)$$

$$= 2 \sum_{j=1}^{m} ((H + K)^{-1} (x_j + y_j)/2^{1/2}, (x_j + y_j)/2^{1/2}).$$

Now clearly $((x_j + y_j)/2^{1/2}, (x_k + y_k)/2^{1/2}) = \delta_{jk}$ so that applying Fan’s result again to the right side of (12) yields (5).

3. An example. Since (1) holds for $p \geq 1$ it is plausible to conjecture that under the same hypotheses as Theorem 2, one has

$$((H^{-p} x, x) + (K^{-p} y, y) \geq ((H + K)^{-p} x + y, x + y),$$

for $p \geq 1$. However, (13) is false in general for $p \geq 1$. In particular let $p = 2$, $y = 0$, $u = (H + K)^{-1} x$ and the statement (13) becomes

$$\|u\|^2 \leq \|(I + H^{-1} K) u\|^2.$$ 

Now (14) is a possibility for all $u$ if and only if the minimum singular value [1, p. 69] of $I + H^{-1} K$ is at least 1. At this point we use the following elementary result:

An $n$-square matrix $A$ is the product of two positive-definite hermitian
matrices if and only if it has positive eigenvalues and linear elementary divisors. For if \( A = PQ \) where \( P \) and \( Q \) are positive-definite then \( P^{-1/2}AP^{1/2} = P^{1/2}QP^{1/2} \) which is conjunctive to \( Q \), and hence has positive eigenvalues and linear elementary divisors. But \( A \) is similar to \( P^{1/2}QP^{1/2} \). Conversely, if \( A \) has linear elementary divisors and positive eigenvalues, then \( A = S^{-1}DS \) in which \( D \) is a diagonal matrix with positive main diagonal entries. Let \( S = UH \) be the polar decomposition of \( S \) so that

\[
A = H^{-1}U^*DUH = H^{-2}H(U^*DU)H.
\]

Then both \( H^{-2} \) and \( H(U^*DU)H \) are positive-definite.

Thus we know for example that the matrix

\[
A = \begin{pmatrix}
1 & 5 \\
0 & 2
\end{pmatrix}
\]

is of the form \( H^{-1}K \) for appropriate positive-definite \( H \) and \( K \). It is elementary to compute that in this case the minimum singular value of \( I + A \) is less than 1 and hence (14) is not true for all \( u \).

We mention that in case \( H \) and \( K \) commute then (13) does hold for \( p \geq 1 \). This is an easy consequence of the fact that \( H \) and \( K \) possess a common orthonormal basis of eigenvectors.

**Reference**


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