ON PRODUCTS OF FINITE DIMENSIONAL STOCHASTIC MATRICES[1]

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1. In what follows, P is the monoid of all $p\times p$ stochastic matrices where p is a fixed natural number. For $a, a'\in P$, we let βa denote the "type" of a [1], i.e. the set of all pairs of indices (j,j') such that the element $a_{j,j'}$ of a is positive and we set $\|a-a'\| = \mathrm{Max}_j \sum_{j'} |a_{j,j'}-a'_{j,j'}|$. Letting ϵ and ω be two fixed positive quantities and $P(\omega)$ be the subset of all $a\in P$ having no positive element less than ω, we intend to verify the following partial generalization of a theorem of Wolfowitz [1].

PROPERTY. *There exists a natural number ν_* such that any product of more than ν_* matrices of $P(\omega)$ admits at least one nontrivial subproduct a which satisfies*

$$\beta a = \beta a^2 \quad \textit{and} \quad \sup_{n,n'\in N} \|a^{1+n} - a^{1+n'}\| \leqq \epsilon.$$

Our number ν_* is quite extravagant and examples such as $\lim_{n\to\infty} \prod_{0\leqq i<n}(x^{m_i}y)$ where the integers m_i grow fast enough,

$$x = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

indicate that little information on an infinite product of stochastic matrices is gained when knowing that for each positive ϵ it admits an infinity of subproducts a $(=x^{m_i})$ satisfying the relations stated in the Property (cf. [2]).

I am most indebted to Professor J. Wolfowitz for many suggestions and advice which have led to the writing of this note.

2. **Verification of the property.** We say that two products $x_1x_2 \cdots x_n$ and $x'_1x'_2 \cdots x'_{n'}$ of matrices x_i, x'_i are *β-equivalent* iff $n=n'$ and $\beta x_i=\beta x'_i$ for $i=1, 2, \cdots, n$. They are *nontrivial* iff $n>0$.

Let $q=(2^p-1)^p$ $(=\mathrm{Card}\ \beta P)$ and define inductively a map

Received by the editors September 9, 1966.

[1] Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of the RAND Corporation or the official opinion or policy of any of its governmental or private research sponsors.

This paper has been sponsored in part by the U. S. Air Force under contract AF61(052)-945.

$\nu: \boldsymbol{N} \to \boldsymbol{N}$ and a sequence $(n_0, n_1, \cdots, n_q)$ of natural numbers by the following conditions:

$\nu(0)=1$; for each $n \in \boldsymbol{N}$, $\nu(n+1)=(1+q^{\nu(n)})\cdot\nu(n)$.

$n_0=1$; for each $i \in N$, $n_{i+1}=1+n_i+$the least positive number m such that $1-\omega^{\nu(n_i)}$ to the power 2^m-1 is $\leqq \epsilon/20$.

REMARK 1. *Any product of $\nu_*=\nu(n_q)$ matrices of $P(\omega)$ admits a subproduct $a=h_1h_1'h_2h_2'h_3 \cdots h_{2s+1}h_{2s+1}'$ where*

(i) *all the $2s+1$ subproducts h_i are β-equivalent products of s' matrices of $P(\omega)$ and $(1-\omega^{s'})^s \leqq \epsilon/20$;*

(ii) *$\beta h_1=\beta a \cdot \beta h_1$ and $\beta a=\beta a^2$.*

PROOF. Call 0-*sesquipower* any nontrivial product and, inductively, say that a product is a $(n+1)$-*sesquipower* iff it has the form $hh'h''$ where h' is arbitrary and where h and h'' are β-equivalent n-sesquipowers.

We verify first that any product of $\nu(n)$ matrices admits at least one n-sesquipower as a subproduct.

Indeed, it is trivial for $n=0$ since $\nu(0)=1$. If it is true for n and if f is a product of $\nu(n+1)$ matrices, the definition of ν implies that $f=f_1f_2 \cdots f_{\bar{q}}$ $(\bar{q}=1+q^{\nu(n)})$ where each f_i is a product of $\nu(n)$ matrices. Since $q^{\nu(n)}$ is precisely the number of classes of β-equivalent product of $\nu(n)$ matrices, at least two of these subproducts (say f_j and $f_{j'}$) are β-equivalent. By the induction hypothesis we have $f_j=g_jhg_j'$; $f_{j'}=g_{j'}h''g_{j'}'$ where h and h'' are β-equivalent n-sesquipower and the statement is verified since f admits the subproduct $hh'h''$ where $h'=g_j'f_{j+1}f_{j+2} \cdots f_{j'-1}g_{j'}$.

In particular, if f is a product of ν_* matrices it admits a n_q-sesquipower k_q as a subproduct and for $j=q-1, q-2, \cdots, 0$; k_q admits a n_j-sesquipower k_j as a right subproduct. Again because of $q=\text{Card}\,\beta P$, at least two of these products (say $k_{j'}$ and k_j) have the same type. We can write $k_{j'}=h_1h_1'h_2 \cdots h_{2s+1}h_{2s+1}'k_j$ where $2s+2=2^{n_{j'}-n_j}$ and where all the subproducts h_i are β-equivalent to k_j. The conditions (i) of the Remark are automatically satisfied because of our choice of the subsequence (n_j) when we take $a=h_1h_1'h_2 \cdots h_{2s+1}h_{2s+1}'$ and we have $\beta h_1=\beta a \cdot \beta h_1$ since h_1, k_j and $k_{j'}=ak_j$ have the same image by β. The equation $\beta a=\beta a^2$ follows instantly from $\beta k_{j'}=\beta k_j=\beta h_1$ when multiplying on the right $k_{j'}=ak_j$ by $h_1'h_2 \cdots h_{2s+1}h_{2s+1}'$. The remark is verified.

REMARK 2. *For each given natural number n there exist a nonnegative quantity $\epsilon' \leqq \epsilon/10$ and two matrices $d, d' \in P$ that satisfy $a^{1+n}=(1-\epsilon')\cdot d+\epsilon'\cdot d'$ and $d=d\bar{a}d$ where $\bar{a}=\lim_{m\to\infty} a^m$.*

PROOF. We identify the indices $1, 2, \cdots, p$ with the states of the

Markov chain defined by the matrix a and we suppose that it has r ergodic classes $E_1, E_2, \cdots, E_r$.

For each $x \in P$, let

$$\pi x = \text{Inf}\{\eta \in [0, 1]: x = (1 - \eta)\cdot x_r + \eta \cdot x_p; x_r \in \overline{P}_r; x_p \in P\}$$

where $\overline{P}_r$ (resp. $\overline{P}_p$) denotes the convex closure of the set P_r (resp. P_p) of all matrices $y \in P$ having entries 0 or 1 only and at most r (resp. p) nonzero columns. Thus, unless $\pi x = 1$, we have $1 - \pi x \geqq$ the least positive entry of x. Further, $\pi(xx') \leqq \pi x \cdot \pi x'$ for any $x, x' \in P$ since $P = \overline{P}_p$ and $P_r \subseteq P_p P_r P_p$ (cf. [3]).

In particular, $\pi a < 1$ because the relation $\beta a = \beta a^2$ implies that the type of any row of a contains at least one of the r ergodic classes. Taking $\beta k_{j'} = \beta k_j = \beta a \cdot \beta k_j$ into account, we deduce $\pi k_j < 1$, hence $\pi h_i < 1 - \omega^{s'}$ ($i = 1, 2, \cdots, 2s+1$) since each h_i is a product of s' matrices of $P(\omega)$ that is β-equivalent to k_j.

Let us now define $b = a^n h_1 h_1' \cdots h_s h_s'$; $c = h_{s+1} h_{s+1}' \cdots h_{2s+1} h_{2s+1}'$; $d = b_r c_r$ ($b_r, c_r \in \overline{P}_r$); $\epsilon' = \pi b + \pi c - \pi b \cdot \pi c$. Because of the submultiplicative character of the map π and $(1 - \omega^{s'})^s \leqq \epsilon/20$, we have $\pi b, \pi c \leqq \epsilon/20$, hence $\epsilon' \leqq \epsilon/10$ and $a^{1+n} = (1 - \epsilon')\cdot d + \epsilon' \cdot d'$ for a suitable $d' \in P$. Further, $\beta d \subseteq \beta a^{1+n} = \beta a$ and $\beta(d\bar{a}d) \subseteq \beta a$ because of $\beta \bar{a} \subseteq \beta a$. Since d and $d\bar{a}d$ are stochastic matrices, this shows that they have at least r characteristic roots equal to 1. To verify $d = d\bar{a}d$ we have only to check that the dimension of the null space of d has its maximal value $p - r$, since then it will follow that d and $d\bar{a}d$ are two commuting idempotent matrices having the same rank.

Consider any index i belonging to some ergodic class $E_{r'}$ ($1 \leqq r' \leqq r$). Since $\beta(b_r c_r) \subseteq \beta a$ and since $E_{r'}$ is precisely the type of the ith row of a, we see that the type of the ith row of b_r must be contained in the set $E'_{r'}$ of the indices i' such that the type of the i'th row of c_r is contained in $E_{r'}$. The r sets $E'_{r'}$ are pairwise disjoint. Thus, since $b_r, c_r \in \overline{P}_r$, on the one hand the index of any nonzero column of b_r belongs to $\bigcup\{E'_{r'}: 1 \leqq r' \leqq r\}$ and, on the other hand, the type of any nonzero column of each $y \in P_r$ satisfying $\beta y \subseteq \beta c_r$ contains one (and only one) of the sets $E'_{r'}$.

Let $e'_{r'}$ be the diagonal matrix such that for any $j, j' = 1, 2, \cdots, p$, its (j, j') entry is equal to 1 or to 0 depending upon $j = j' \in E'_{r'}$ or not and $e' = e'_1 + e'_2 + \cdots + e'_r$. The first statement above implies that $d = b_r c_r = b_r e' c_r$ while the second one shows that all the nonzero rows of each matrix $e'_{r'} c_r$ are equal, hence that the null space of $e' c_r$ has dimension at least $p - r$. Remark 2 is verified.

Substituting $(1 - \epsilon')\cdot d + \epsilon' \cdot d'$ for a^{1+n} in the right member of

$a^{1+n}-\bar{a}=a^{1+n}-a^{1+n}\bar{a}a^{1+n}$ and recalling that $\|x\|=1$ for any $x\in P$, we obtain

$$\begin{aligned}\|a^{1+n}-\bar{a}\| &= \epsilon'\cdot\|(1-\epsilon')(d-d\bar{a}d'-d'\bar{a}d)+d'-\epsilon'\cdot d'\bar{a}d'\| \\ &\leqq 5\epsilon' \leqq \epsilon/2.\end{aligned}$$

In view of the triangular inequality, this concludes the verification of the Property.

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