ON PRODUCTS OF FINITE DIMENSIONAL
STOCHASTIC MATRICES

M. P. SCHÜTZENBERGER

1. In what follows, $P$ is the monoid of all $p \times p$ stochastic matrices
where $p$ is a fixed natural number. For $a, a' \in P$, we let $\beta a$ denote
the "type" of $a$ [1], i.e. the set of all pairs of indices $(j, j')$ such that
the element $a_{j, j'}$ of $a$ is positive and we set $\|a - a'\| = \max_j \sum_{j'} |a_{j,j'} - a'_{j,j'}|$. Letting $\varepsilon$ and $\omega$ be
two fixed positive quantities and $P(\omega)$ be the subset
of all $a \in P$ having no positive element less than $\omega$, we intend to verify
the following partial generalization of a theorem of Wolfowitz [1].

**Property.** There exists a natural number $\nu_*$ such that any product of
more than $\nu_*$ matrices of $P(\omega)$ admits at least one nontrivial subproduct
which satisfies

$$\beta a = \beta a^2 \quad \text{and} \quad \sup_{n, n' \in \mathbb{N}} \|a^{1+n} - a^{1+n'}\| \leq \varepsilon.$$ 

Our number $\nu_*$ is quite extravagant and examples such as
$$\lim_{n \to \infty} \prod_{0 \leq i < n} (x^{m_i}y)$$
where the integers $m_i$ grow fast enough,

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

indicate that little information on an infinite product of stochastic
matrices is gained when knowing that for each positive $\varepsilon$ it admits
an infinity of subproducts $a (= x^{m_i})$ satisfying the relations stated in
the Property (cf. [2]).

I am most indebted to Professor J. Wolfowitz for many suggestions
and advice which have led to the writing of this note.

2. Verification of the property. We say that two products $x_1x_2 \cdots x_n$
and $x'_1x'_2 \cdots x'_n$ of matrices $x_i, x'_i$ are $\beta$-equivalent iff $n = n'$ and
$\beta x_i = \beta x'_i$ for $i = 1, 2, \cdots, n$. They are nontrivial iff $n > 0$.

Let $q = (2^p - 1)^p$ ($= \text{Card } \beta P$) and define inductively a map

Received by the editors September 9, 1966.

1 Any views expressed in this paper are those of the author. They should not be interpreted as reflecting the views of the RAND
Corporation or the official opinion or policy of any of its governmental or private research sponsors.

This paper has been sponsored in part by the U. S. Air Force under contract,
AF61(052)-945.
\( \nu: \mathbb{N} \rightarrow \mathbb{N} \) and a sequence \((n_0, n_1, \cdots, n_\varphi)\) of natural numbers by the following conditions:

\( \nu(0) = 1 \); for each \( n \in \mathbb{N} \), \( \nu(n+1) = (1 + q^{(n)}) \cdot \nu(n) \).

\( n_0 = 1 \); for each \( i \in \mathbb{N} \), \( n_{i+1} = 1 + n_i + \text{the least positive number } m \) such that \( 1 - \omega^{(n)} \) to the power \( 2^m - 1 \) is \( \leq \varepsilon/20 \).

**Remark 1.** Any product of \( \nu \circ \nu \circ \cdots \circ \nu \) matrices of \( \mathcal{P}(\omega) \) admits a subproduct \( a = h_1 h'_1 h_2 h'_2 \cdots h_{2s+1} h'_{2s+1} \) where

(i) all the \( 2s+1 \) subproducts \( h_i \) are \( \beta \)-equivalent products of \( s' \) matrices of \( \mathcal{P}(\omega) \) and \( (1 - \omega^s)^s \leq \varepsilon/20 \);

(ii) \( \beta h_1 = \beta a \cdot \beta h_1 \) and \( \beta a = \beta a^2 \).

**Proof.** Call 0-\( \text{sesquipower} \) any nontrivial product and, inductively, say that a product is a \((n+1)-\text{sesquipower} \) iff it has the form \( hh' h'' \) where \( h' \) is arbitrary and where \( h \) and \( h'' \) are \( \beta \)-equivalent \( n \)-sesquipowers.

We verify first that any product of \( \nu(n) \) matrices admits at least one \( n \)-sesquipower as a subproduct.

Indeed, it is trivial for \( n = 0 \) since \( \nu(0) = 1 \). If it is true for \( n \) and if \( f \) is a product of \( \nu(n+1) \) matrices, the definition of \( \nu \) implies that \( f = f_1 f_2 \cdots f_\varphi \) \((\varphi = 1 + q^{(n)})\) where each \( f_i \) is a product of \( \nu(n) \) matrices. Since \( q^{(n)} \) is precisely the number of classes of \( \beta \)-equivalent product of \( \nu(n) \) matrices, at least two of these subproducts (say \( f_i \) and \( f_j \)) are \( \beta \)-equivalent. By the induction hypothesis we have \( f_j = g_j h g_j' \); \( f_j' = g_j' h'' g_j' \) where \( h \) and \( h'' \) are \( \beta \)-equivalent \( n \)-sesquipower and the statement is verified since \( f \) admits the subproduct \( hh' h'' \) where \( h = g_j f_{j+1} f_{j+2} \cdots f_{\varphi-1} g_j' \).

In particular, if \( f \) is a product of \( \nu \) matrices it admits a \( n_q \)-sesquipower \( k_q \) as a subproduct and for \( j = q - 1, q - 2, \cdots, 0 \); \( k_q \) admits a \( n_j \)-sesquipower \( k_j \) as a right subproduct. Again because of \( q = \text{Card } \beta P \), at least two of these products (say \( k_j' \) and \( k_j \)) have the same type. We can write \( k_j' = h_1 h'_1 h_2 \cdots h_{2s+1} h'_{2s+1} k_j \) where \( 2s+2 = 2n_j - nj \) and where all the subproducts \( h_i \) are \( \beta \)-equivalent to \( k_j \). The conditions (i) of the Remark are automatically satisfied because of our choice of the subsequence \((n_j)\) when we take \( a = h_1 h'_1 h_2 \cdots h_{2s+1} h'_{2s+1} \) and we have \( \beta h_i = \beta a \cdot \beta h_i \) since \( h_1, k_j \) and \( k_j' = ak_j \) have the same image by \( \beta \). The equation \( \beta a = \beta a^2 \) follows instantly from \( \beta k_j' = \beta k_j = \beta h_1 \) when multiplying on the right \( k_j' = ak_j \) by \( h'_1 h_2 \cdots h_{2s+1} h'_{2s+1} \). The remark is verified.

**Remark 2.** For each given natural number \( n \) there exist a nonnegative quantity \( \varepsilon' \leq \varepsilon/10 \) and two matrices \( d, d' \in \mathcal{P} \) that satisfy \( a^{1+n} = (1 - \varepsilon') \cdot d + \varepsilon' \cdot d' \) and \( d = d a d \) where \( a = \lim_{m \rightarrow \infty} a^m \).

**Proof.** We identify the indices 1, 2, \cdots, \( p \) with the states of the
Markov chain defined by the matrix $a$ and we suppose that it has $r$

ergodic classes $E_1, E_2, \cdots, E_r$.

For each $x \in P$, let

$$\pi x = \inf\{\eta \in [0, 1] : x = (1 - \eta) x_r + \eta x_p, x_r \in \overline{P}_r; x_p \in P\}$$

where $\overline{P}_r$ (resp. $\overline{P}_p$) denotes the convex closure of the set $P_r$ (resp. $P_p$) of all matrices $y \in P$ having entries 0 or 1 only and at most $r$ (resp. $p$) nonzero columns. Thus, unless $\pi x = 1$, we have $1 - \pi x \geq$ the least positive entry of $x$. Further, $\pi(xx') \leq \pi x \cdot \pi x'$ for any $x, x' \in P$ since $P = \overline{P}_p$ and $P_r \subseteq P_p P_r P_p$ (cf. [3]).

In particular, $\pi a < 1$ because the relation $\beta a = \beta a^2$ implies that the type of any row of $a$ contains at least one of the $r$ ergodic classes. Taking $\beta k_i = \beta k_j = \beta a \cdot \beta k_i$ into account, we deduce $\pi k_i < 1$, hence $\pi x_i < 1 - \omega^r$ $(i = 1, 2, \cdots, 2s + 1)$ since each $h_i$ is a product of $s'$ matrices of $P(\omega)$ that is $\beta$-equivalent to $k_j$.

Let us now define $b = a^n h_1 h_2 \cdots h_s h_{s'}$; $c = h_{s+1} h_{s+2} \cdots h_{2s+1} h_{2s'+1}$; $d = b, c, (b_r, c_r) \in \overline{P}_r$; $e' = \pi b + \pi c - \pi b \cdot \pi c$. Because of the submultiplicative character of the map $\pi$ and $(1 - \omega^r)^{s} \leq \epsilon/20$, we have $\pi b, \pi c \leq \epsilon/20$, hence $e' \leq \epsilon/10$ and $a^{1+n} = (1 - \epsilon') \cdot d + e' \cdot d'$ for a suitable $d' \in P$. Further, $\beta d \subseteq \beta a^{1+n} = \beta a$ and $\beta (d a, d a) \subseteq \beta a$ because of $\beta a \subseteq \beta a$. Since $d$ and $d a, d a$ are stochastic matrices, this shows that they have at least $r$ characteristic roots equal to 1. To verify $d = d a, d a$ we have only to check that the dimension of the null space of $d$ has its maximal value $p - r$, since then it will follow that $d$ and $d a, d a$ are two commuting idempotent matrices having the same rank.

Consider any index $i$ belonging to some ergodic class $E_{r'}$ $(1 \leq r' \leq r)$.

Since $\beta (b, c,) \subseteq \beta a$ and since $E_{r'}$ is precisely the type of the $i$th row of $a$, we see that the type of the $i$th row of $b$ must be contained in the set $E_{r'}$ of the indices $i'$ such that the type of the $i'$th row of $c_r$ is contained in $E_{r'}$. The $r$ sets $E_{r'}$ are pairwise disjoint. Thus, since $b, c_r \in \overline{P}_r$, on the one hand the index of any nonzero column of $b$ belongs to $\bigcup E_{r'} : 1 \leq r' \leq r$ and, on the other hand, the type of any nonzero column of each $y \in P$, satisfying $\beta y \subseteq \beta c_r$, contains one (and only one) of the sets $E_{r'}$.

Let $e'_r$ be the diagonal matrix such that for any $j, j' = 1, 2, \cdots, p$, its $(j, j')$ entry is equal to 1 or to 0 depending upon $j = j' \in E_{r'}$ or not and $e' = e'_1 + e'_2 + \cdots + e'$. The first statement above implies that $d = b, c_r = b, e'_1 c_r$ while the second one shows that all the nonzero rows of each matrix $e'_r c_r$ are equal, hence that the null space of $e' c_r$ has dimension at least $p - r$. Remark 2 is verified.

Substituting $(1 - e') \cdot d + e' \cdot d'$ for $a^{1+n}$ in the right member of
\[ a^{1+n} - \tilde{a} = a^{1+n} - a^{1+n} \tilde{a} a^{1+n} \] and recalling that \( \| x \| = 1 \) for any \( x \in P \), we obtain
\[
\| a^{1+n} - \tilde{a} \| = \epsilon' \cdot \| (1 - \epsilon')(\tilde{d} - \tilde{d}' \tilde{a} \tilde{d} - d' \tilde{a} d) + d' - \epsilon' \cdot d' \tilde{a} \tilde{d} \| \\
\leq 5 \epsilon' \leq \epsilon/2.
\]

In view of the triangular inequality, this concludes the verification of the Property.

**References**


Faculté des Sciences, Paris and RAND Corporation