THE SHARPENING OF A RESULT CONCERNING
PRIMITIVE IDEALS OF AN ASSOCIATIVE RING

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The importance of the concept of primitive ideals of associative rings consists in the well-known theorem stating that every semisimple ring $A$ is a subdirect sum of primitive rings $B_v$, where a ring $A$ is called semisimple (in the sense of Jacobson) if the Jacobson radical, i.e. the intersection of all primitive ideals, coincides with the zero ideal $(0)$, and a ring $B_v$ is called primitive if the ideal $(0)$ is a primitive ideal of $B_v$. (Cf. N. Jacobson, Structure of rings, Colloq. Publ., Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.)

Some new characterizations were recently given for the Jacobson radical of a ring $A$. For instance, A. Kertész [3] has shown in these Proceedings (generalizing an observation of L. Fuchs [1]) that the Jacobson radical $J$ of a ring $A$ consists of exactly those elements $x$ of $A$ for which the product $yx$ lies with every $y \in A$ in the Frattini $A$-submodule of the ring $A$, as of an $A$-right module $A$ for itself (cf. also Hille [2]). Furthermore A. Kertész [4] has shown that $J$ is the intersection of all those maximal right ideals $R$ of $A$ for which there must exist, for any element $x \in R$ ($x \in A$), a second element $y \in A$ with $yx \in R$; that is, those right ideals for which $A^{-1}R \subseteq R$ holds, where $X^{-1}R = \{y; y \in A, Xy \subseteq R\}$ for an arbitrary subset $X$ of $A$. Furthermore, let $L \cdot Y^{-1}$ denote the subset $\{z; z \in A, zY \subseteq L\}$.

Every modular right ideal $R$ of $A$ is quasi-modular in the sense that $A^{-1}R \subseteq R$ holds. The concept of quasi-modularity of right ideals $R$ was introduced in [6]. Solving a problem proposed by Kertész [4] I have shown in [6] the existence of an associative ring which has a quasi-modular maximal but not a modular right ideal. In my other paper [7] a two-sided ideal $Q$ of $A$ is called quasi-primitive if there exists a quasi-modular maximal right ideal $R$ of $A$ with $Q = A^{-1}R \subseteq R$. Obviously every primitive ideal is also quasi-modular in $A$, and almost trivially every artin ring with $(0)$ quasi-primitive ideal is a total matrix ring over a skew field. Furthermore, any quasi-primitive ideal is clearly a prime ideal, and any commutative ring with $(0)$ quasi-primitive ideal is a field.

Solving a problem of my colleague Dr. Steinfeld, I have proved in [7] that the Jacobson radical $J$ of $A$ must coincide with the intersec-

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tion of all quasi-primitive ideals. There are two proofs of this fact in [7], an entirely elementary proof without quasi-regular element and irreducible modules, and (in a footnote) a second short proof with quasi-regular elements too.

In my note [7] some open problems on quasi-primitive ideals are mentioned, which have recently been solved completely by Dr. Steinfeld. He has shown that the concepts of primitivity and quasi-primitivity of ideals of arbitrary associative rings must coincide. Using a lemma which is proved but not explicitly announced in [7], Dr. Steinfeld has proved that there exists for every fixed quasi-modular maximal right ideal \( R \) of \( A \) an element \( X \) of \( A \) for which the right ideal quotient \( R_x = \{ x \}^{-1} R \) is a modular maximal right ideal of \( A \) such that \( A^{-1} R = A^{-1} R_x = (xA)^{-1} R \), which means that every quasi-primitive ideal \( Q = A^{-1} R \) is by \( Q = A^{-1} R_x \) also primitive in \( A \).

This result of Dr. Steinfeld can be sharpened as follows:

**Theorem.** If \( R \) is a quasi-modular maximal right ideal of an arbitrary associative ring, and if \( x \in A \) is an arbitrary element of \( A \) with the condition \( x \in R \), then the quasi-primitive ideal \( Q = A^{-1} R \) coincides with the primitive ideal \( P_x = A^{-1} R_x = (xA)^{-1} R \) of \( A \) (instead of a single \( x \) for any \( x \in R \)).

**Proof.** In my note [7] it is shown that \( R_x = \{ x \}^{-1} R \) is a modular maximal right ideal of \( A \) for every quasi-modular maximal right ideal \( R \) of \( A \) and for every \( x \in A \) with \( x \in R \). Namely, \( R_x = \{ x \}^{-1} R \) is a right ideal of \( A \). By the quasi-modularity of \( R \), \( A^2 + R = A \), and therefore we obtain \( RA^{-1} = R \); that is, \( xA + R = A \) for any \( x \in R \), \( x \in A \). Since there exists for \( x \in R \) an element \( y \in A \) with \( xy \in R \), the right ideal \( R_x \) has the property \( y \in R_x \), i.e. \( R_x \neq A \). If \( z \in A \) is any element with \( z \in R_x \), one has by \( xz \in R \) obviously \( xzA + R = A \), and thus for any \( b \in A \) the existence of \( a \in A \) and \( r \in R \) with \( xza + r = xb \), and thereby also \( x(b - za) = r \in R \), \( b - za \in R_x \), \( b \in zA + R_x \) and \( A = zA + R_z \), which means the maximality of \( R_z \) in \( A \). Moreover, one has \( xza_1 + r_1 = x \) with some \( a_1 \in A \) and \( r_1 \in R \), which implies \( x(1 - za_1)A \subseteq R \), consequently \( (1 - za_1)A \subseteq R_z \) and the modularity of the maximal right ideal \( R_z \) of \( A \).

By \( xA + R = A \) and \( A((xA)^{-1} R) = (xA + R)((xA)^{-1} R) \subseteq R \) one has on one side \( (xA)^{-1} R \subseteq A^{-1} R \). On the other hand the condition \( y \in A^{-1} R \) implies by \( A^{-1} R = (xA + R)^{-1} R \) obviously \( xAy \subseteq R \), that is \( y \in (xA)^{-1} R \), and thus holds \( A^{-1} R = (xA)^{-1} R \) for every \( x \in R \) (\( x \in A \)). But one has almost trivially \( (xA)^{-1} R = A^{-1} \{ x \}^{-1} R = A^{-1} R_x \) too, which means that \( Q = A^{-1} R = (xA)^{-1} R \) and \( P_x = A^{-1} R_x \) must be for every \( x \in R \) the same primitive ideals of \( A \). Q.E.D.
References


