ON THE ANALOG OF LITTLEWOOD'S PROBLEM
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If $K$ is any field, we can form the field $\mathbb{B}_K$ whose nonzero elements are expressions

$$Z = \sum_{i=0}^{\infty} a_{i-n} n^{-i}$$

with $a_{i-n} \in K$, $a_{-n} \neq 0$. We let $Z'$ denote the sum of those terms of $Z$ for which $i > n$. Following [3], we write $|Z| = e^n$, $|0| = 0$ and $||Z|| = |Z'|$. The main result of [3] is: If $K$ is infinite then there are $\theta$, $\Phi \in \mathbb{B}_K$ such that

$$||N\theta|| \leq e^{-2}$$

for all polynomials $N \neq 0$. Baker [2] has given definite elements of $\mathbb{B}_K$ for which the left side of (1) is bounded below, but his bound is slightly smaller than $e^{-2}$. We now prove an extension of the theorem of Davenport and Lewis. The present method could also be specialized to give a new proof of their result. First, we prove some lemmas about diophantine approximation in $\mathbb{B}_K$.

Let $M = M(d; c_1, \cdots, c_m)$ be the collection of polynomials $N$ which satisfy

$$||N|| \leq e^d, \quad ||N\theta_i|| \leq e^{c_i} \quad (i = 1, \cdots, m)$$

where $d \geq 0$, $c_i \leq -1$. Let $b = \max (0, d+m+1+c_1+\cdots+c_m)$.

**Lemma 1.** $M$ is a $K$-vector space of dimension at least $b$ and at most $d+1$.

**Lemma 2.** $M(d; \cdots) = M(d-1; \cdots) + M'$ where $M'$ is either $\{0\}$ or a 1-dimensional space generated by a polynomial of degree $d$.

**Proof.** When all $c_i = -1$, (2) defines the $(d+1)$-dimensional space of all polynomials of degree at most $d$. Thus Lemma 1 holds in this case. If either $d$ or one of the $c_i$ is reduced by 1, a certain coefficient of $N$ or $N\theta_i$ is required to be 0. This defines a linear subspace of codimension at most 1 in $M$. This observation both proves Lemma 2 and provides the inductive step to prove Lemma 1.

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Remark. This proof allows us to construct spaces defined by the inequalities (2) which are 1-dimensional. When \( m = 1 \) the generators of those spaces can be obtained by a continued fraction (cf. [1, §12]).

Lemma 3. The spaces \( M \) all have exactly the dimension \( b \) (of Lemma 1) if and only if

\[
(3) \quad |N| \cdot \prod_{i=1}^{m} \|N\theta_i\| \geq e^{-m}
\]

for all polynomials \( N \neq 0 \).

Proof. Let \( N \neq 0 \) be a polynomial. The smallest \( M \) containing \( N \) (denoted \( M(N) \)) has \( b \) such that \( e^b = |N| \cdot \prod_{i=1}^{m} \|N\theta_i\| \cdot e^{m+1} \). Also \( \dim M(N) \geq 1 \) so if it is exactly equal to \( b \) we must have (3). On the other hand, if some \( M \) has dimension greater than \( b \), the reduction of Lemma 2 leads to a 1-dimensional \( M \) for which \( b = 0 \). If \( N \) generates this space, it clearly cannot satisfy (3).

Corollary. If (3) holds, the \( M' \) of Lemma 2 is 1-dimensional whenever \( b > 0 \).

Theorem: If \( K \) is infinite, then there is a sequence \( \theta_1, \theta_2, \ldots \) of elements of \( \mathfrak{B}_K \) such that (3) holds for all polynomials \( N \neq 0 \) and each \( m \geq 0 \). Furthermore, if \( \theta_1, \ldots, \theta_m \) satisfy (3) there is such a sequence that begins with these terms.

Proof. By induction, it suffices to produce \( \theta_{m+1} \) when \( \theta_1, \ldots, \theta_m \) are given, since the case \( m = 0 \) is trivially true.

By the proof of Lemma 3 and the inductive hypothesis,

\[
e^{m+1} \cdot |N| \cdot \prod_{i=1}^{m+1} \|N\theta_i\| = e^b \|N\theta_{m+1}\|
\]

where \( b = \dim M(N) \). Thus we must show that the coefficients of \( t^{-1}, \ldots, t^{-b} \) in \( N\theta_{m+1} \) can not all be zero. These coefficients are linear functions on \( M(N) \), so we have a linear function from \( M(N) \) to \( K^b \) which we must show to be an isomorphism. If a basis is chosen for \( M(N) \) this requires only that a certain determinant be nonzero.

Suppose \( \theta_{m+1} = a_1 t^{-1} + a_2 t^{-2} + \cdots \). If \( N \) has degree \( d \), the rule for multiplying elements of \( \mathfrak{B}_K \) gives the coefficient of \( t^{-i} \) in \( N\theta_{m+1} \) as a linear combination of \( a_i, \cdots, a_{d+i} \). Furthermore, \( a_{d+i} \) must occur.

If \( N_1, \ldots, N_b \) is a basis for \( M \) and \( \gamma_{ij} \) is the coefficient of \( t^{-i} \) in \( N_j \theta_{m+1} \), we write \( \Delta = \det(\gamma_{ij}) \) \((1 \leq i, j \leq b)\). Lemma 2 and the corollary to Lemma 3 tell us that \( M \) has a basis in which \( N_1, \ldots, N_{b-1} \) are a basis for \( M(d-1; \cdots) \) and \( N_b \) has degree \( d \). Thus \( \gamma_{ij} \) will be a linear
combination of $a_1, \cdots, a_{b+d}$ with $a_{b+d}$ occurring only in $\gamma_{bb}$. A closer look tells us that $\Delta(d; \cdots) = a_{b+d} \Delta(d-1, \cdots) + (\text{terms containing } a_1, \cdots, a_{b+d-1})$. There are only finitely many choices of $d, c_1, \cdots, c_m$ for which $b+d = k$. Thus if $a_1, \cdots, a_{k-1}$ have been obtained such that all $\Delta$ with $b+d < k$ are not zero, there are only a finite number of values of $a_k$ which cause any $\Delta$ with $b+d = k$ to vanish. Since $K$ is infinite, we can determine a $\theta_{m+1}$ so that no $\Delta$ is zero. This is all that is required to prove the theorem.

REFERENCES


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