COMMUTATIVE RINGS OVER WHICH EVERY MODULE HAS A MAXIMAL SUBMODULE

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In this note we characterize those commutative rings over which every nonzero module has a maximal submodule. Professor Hyman Bass in [1, p. 470] states the following conjecture: a ring R is left perfect if, and only if, every nonzero left R-module has a maximal submodule, and R has no infinite set of orthogonal idempotents. For commutative rings we also show that Bass' conjecture is true.

Throughout, R will be a ring with identity, J will denote the Jacobson radical of R, and S will denote the ring R/J. We use the word *module* to mean unital module. If M is a left R-module, rad M denotes the radical of M, that is, the intersection of the maximal submodules of M. If $m \in M$, then ${}_R(0: m) = \{r \in R: rm = 0\}$.

A left ideal L in R is *left T-nilpotent* if for each sequence $\{r_i\}_{i=1}^{\infty}$ in L, there is some positive integer k with $r_1 \cdots r_k = 0$. A submodule B of an R-module M is *small in* M if $B + M' = M$ where M' is a submodule of M implies that $M' = M$.

The first lemma occurs as a remark in [1, p. 470].

LEMMA 1. *Every nonzero left R-module has a maximal submodule* $\Leftrightarrow J$ *is left T-nilpotent, and every nonzero left S-module has a maximal submodule.*

PROOF. ($\Rightarrow$) Clearly every nonzero left S-module has a maximal submodule if every nonzero left R-module has a maximal submodule. Suppose that B is a nonzero left R-module, and A is a submodule of B with $A + \text{rad } B = B$. If $A \neq B$, then by our hypothesis B/A has a maximal submodule, that is, there exists a maximal submodule $\bar{A}$ of B which contains A. Since by definition rad $B \subset \bar{A}$, we have that $A + \text{rad } B = B \subset \bar{A}$, a contradiction. Thus $A = B$, and we have shown that for each left R-module B, rad B is small in B.

Now let $\{j_i\}_{i=1}^{\infty}$ be a sequence of elements in J. If F is the free R-module with basis $\{x_i\}_{i=1}^{\infty}$, and F' is the submodule of F generated by $\{x_i - j_i x_{i+1}\}_{i=1}^{\infty}$, then since rad $F = JF$, and consequently $F' + \text{rad } F$

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$=F$, we must have that $F'=F$. Thus there exist $r_1, \cdots, r_n \in R$ with

$$x_1 = \sum_{i=1}^{n} r_i(x_i - j_i x_{i+1})$$
$$= r_1x_1 + \left\{ \sum_{i=2}^{n} (r_i - r_{i-1}j_{i-1})x_i \right\} - r_nj_nx_{n+1}.$$

By the independence of the x_i's, $r_1 = 1$, $r_i = r_{i-1}j_{i-1}$ for $i=2, \cdots, n$, and $r_nj_n = 0$. Now $r_2 = r_1j_1 = j_1$. In general if $r_k = j_1 \cdots j_{k-1}$ where $2 \leq k < n$, then $r_{k+1} = r_kj_k = j_1 \cdots j_k$. By this process we find that $r_n = j_1 \cdots j_{n-1}$, and $j_1 \cdots j_n = r_nj_n = 0$. This shows that J is left T-nilpotent.

($\Leftarrow$) Now assume that J is left T-nilpotent, and that every nonzero left S-module has a maximal submodule. Let M be a nonzero left R-module, and assume that $JM=M$. Suppose that $jm \neq 0$ for $j \in J$ and $m \in M$. Then since $m = \sum_{i=1}^{n} j_im_i$ where $j_i \in J$ and $m_i \in M$, there is a subscript k with $jj_km_k \neq 0$. Now since $JM \neq 0$, there are elements $j_1 \in J$ and $m_1 \in M$ with $j_1m_1 \neq 0$. By the above argument we can produce elements $j_2 \in J$ and $m_2 \in M$ with $j_1j_2m_2 \neq 0$. By induction, there exists a sequence $\{j_i\}_{i=1}^{\infty}$ in J and a sequence $\{m_i\}_{i=1}^{\infty}$ in M with $j_1 \cdots j_km_k \neq 0$ for $k=1, 2, \cdots$. This contradicts the fact that J is left T-nilpotent, and hence for a nonzero module M, $JM \neq M$. Thus M/JM, a nonzero left S-module, has a maximal submodule, and so must M. This completes the proof of Lemma 1.

REMARK. Lemma 1 admits the following generalization: every nonzero left R-module has a maximal submodule $\Leftrightarrow$ for each left R-module M, each sequence of homomorphisms $\{\phi_i\}_{i=1}^{\infty}$ in $\mathrm{Hom}_R(M, \mathrm{rad}\ M)$, and each $m \in M$, there is a positive integer k with $(m)\phi_1 \cdots \phi_k = 0$. The implication ($\Leftarrow$) is trivial since if there is a nonzero left R-module M such that $M = \mathrm{rad}\ M$, then taking ϕ_i to be the identity map of M to itself for $i=1, 2, \cdots$, for any nonzero $m \in M$, $(m)\phi_1 \cdots \phi_k \neq 0$ for each positive integer k. Thus for a nonzero left R-module M, $M \neq \mathrm{rad}\ M$. For the reverse implication ($\Rightarrow$) let M, a left R-module, and $\{\phi_i\}_{i=1}^{\infty}$ in $\mathrm{Hom}_R(M, \mathrm{rad}\ M)$ be given. Then letting $M_i = M$ for $i=1, 2, \cdots$, form the direct system of R-modules $\{M_i\}_{i=1}^{\infty}$ with homomorphisms ϕ_i: $M_i \to M_{i+1}$. Let L be the direct limit, and for $i=1, 2, \cdots$, let ψ_i: $M_i \to L$ be the induced homomorphism. If A is a maximal submodule of L, and $(M_i)\psi_i \not\subset A$, then $\{A \cap (M_{i+1})\psi_{i+1}\}\psi_{i+1}^{-1}$ is a maximal submodule of M_{i+1} which does does not contain $(M_i)\phi_i$, a contradiction. Thus L has no maximal submodule. We conclude that $L \approx 0$ which implies that given $m \in M$ there is a positive integer k with $(m)\phi_1 \cdots \phi_k = 0$.

LEMMA 2. *If R is commutative, and every nonzero R-module has a maximal submodule, then every element of R which is not a zero divisor is a unit.*

PROOF. Let x be an element of R which is not a zero divisor. Let $A=\sum_{i=1}^{\infty}\oplus Ry_i$ where $Ry_i \approx R/Rx^i$, that is, ${}_R(0\colon y_i)=Rx^i$, and let $B=\sum_{i=1}^{\infty}R(xy_{i+1}-y_i)$. Then $A/B=\sum_{i=1}^{\infty}R\bar{y}_i$ where $\bar{y}_i=y_i+B$. Suppose that $A\neq B$; then by hypothesis A/B has a maximal submodule M. If $\bar{y}_n \notin M$, then there exist $r\in R$ and $m\in M$ such that $r\bar{y}_n+m=\bar{y}_{2n}$. But, using the commutativity of R, we then have $\bar{y}_n=x^n\bar{y}_{2n}=rx^n\bar{y}_n+x^nm=x^nm\in M$, a contradiction. Hence $\bar{y}_n\in M$ for $n=1, 2, \cdots$, and $M=A/B$. This contradiction shows that $A=B$ so there are $r_1, \cdots, r_n\in R$ with

$$y_1 = \sum_{i=1}^{n} r_i(xy_{i+1} - y_i)$$

$$= - r_1y_1 + \left\{ \sum_{i=2}^{n} (r_{i-1}x - r_i)y_i \right\} + r_nxy_{n+1}.$$

Since the y_i's are independent, $y_1=-r_1y_1$, $r_{i-1}x-r_i\in{}_R(0\colon y_i)=Rx^i$ for $i=2, \cdots, n$, and $r_nx\in{}_R(0\colon y_{n+1})=Rx^{n+1}$. Since $r_nx\in Rx^{n+1}$, and x is not a zero divisor, $r_n\in Rx^n$. Suppose that $r_k\in Rx^k$ where $2\leqq k\leqq n$. Since $r_{k-1}x-r_k\in Rx^k$, $r_{k-1}x\in Rx^k$. As x is not a zero divisor, $r_{k-1}\in Rx^{k-1}$. This finite induction shows that $r_1\in Rx$. Then $y_1=-r_1y_1=0$ so $R/Rx\approx Ry_1\approx 0$. Hence x is a unit.

THEOREM. *Let R be a commutative ring. Then every nonzero R-module has a maximal submodule$\Leftrightarrow J$ is T-nilpotent, and S is a (von Neumann-) regular ring.*

PROOF. ($\Leftarrow$) To demonstrate this implication, it suffices by Lemma 1 to show that every nonzero S-module has a maximal submodule. Since S is regular and commutative, by a result of Professor Irving Kaplansky [**5**, Theorem 6, p. 380] every simple S-module is injective. Let M be a nonzero S-module, and let $m\in M$, $m\neq 0$. Since Sm is cyclic, there is an epimorphism ψ from Sm to a simple S-module A. If $i\colon Sm\rightarrow M$ is the identity injection, then there is a homomorphism $\phi\colon M\rightarrow A$ with $i\phi=\psi$. Thus ϕ is an epimorphism, and Ker ϕ is a maximal submodule of M.

($\Rightarrow$) By Lemma 1 J is left T-nilpotent, and every nonzero S-module has a maximal submodule. Let $a\in S$, $a\neq 0$. Since S is commutative and has no nilpotent ideals, $Sa\cap{}_S(0\colon a)=0$. Let $\bar{S}=S/{}_S(0\colon a)$, and for $s\in S$ let $\bar{s}=s+{}_S(0\colon a)$. If $\bar{s}\bar{a}=\bar{0}$, then $sa\in Sa\cap{}_S(0\colon a)=0$ so

$s \in {}_S(0: a)$, and $\bar{s} = \bar{0}$. Thus $\bar{a}$ is an element of $\bar{S}$ which is not a zero divisor. Since $\bar{S}$ is commutative, and every nonzero $\bar{S}$-module has a maximal submodule, by Lemma 2 $\bar{S}\bar{a} = \bar{S}$, and we then have that $Sa \oplus {}_S(0: a) = S$. This shows that every principal ideal of S is a direct summand of S so S is regular.

COROLLARY. *A commutative ring R is perfect $\Leftrightarrow$ every nonzero R-module has a maximal submodule, and R has no infinite set of orthogonal idempotents.*

PROOF. ($\Rightarrow$) This follows from the results in [**1**].

($\Leftarrow$) The theorem states that J is T-nilpotent, and R/J is regular. Since R is an SBI-ring [**3**, Proposition 3, p. 54 and Remark, p. 55], countable sets of orthogonal idempotents in R/J lift orthogonally to R. Thus R/J has no infinite set of orthogonal idempotents, and hence R/J is semisimple artin. By [**1**, Theorem P, p. 467] R is perfect.

No internal characterization of rings all of whose nonzero left modules have maximal submodules is known. Lemma 1 reduces this problem to the case of semisimple rings. In this connection consider the following three properties on a ring R: (i) R is semisimple, and every nonzero left R-module has a maximal submodule; (ii) every simple left R-module is injective; and (iii) R is (von Neumann-) regular. By Professor Kaplansky's result [**5**, Theorem 6, p. 380] and the theorem, for commutative rings (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii). In general the status of only two of the six possible implications is known. (ii) is equivalent to the condition that every left R-module has zero radical, so in general (ii)$\Rightarrow$(i).

In [**4**] it is shown that if $\aleph$ is a transfinite cardinal number, and if V_F is an $\aleph$-dimensional vector space over a field F whose cardinality does not exceed $2^{\aleph}$, then L, the full ring of linear transformations on V_F, is regular but possesses both right and left simple modules which are not injective. Thus (iii) does not imply (ii). It would be interesting to know, and is an open question, whether over a ring L as above, there is a nonzero right or left L-module which has no maximal submodule.

In [**2**] it is shown that if R is a commutative, noetherian ring, then every nonzero R-module has a maximal submodule if, and only if, R is a test module for projectivity. The results here do not seem to shed any new light on the relationship of the two conditions for commutative rings.

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