EXISTENCE CONDITIONS FOR THE LANE INTEGRAL

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1. **Introduction.** We deal in this paper with an integral which was defined by R. E. Lane [4] and which is an extension of the Stieltjes mean sigma integral introduced by H. L. Smith [6]. If f and g are real-valued functions on an interval $[a, b]$ of the real axis, and if $D=\{a=x_0<x_1<x_2< \cdots <x_n=b\}$ is a subdivision of $[a, b]$, we use the symbol $S_D(f, g)$ to denote the sum

$$\sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2} \cdot [g(x_i) - g(x_{i-1})].$$

If $[a, b]$ is an interval of the real axis, and if G is a subset of $[a, b]$ containing a and b, we use the terminology G-subdivision of $[a, b]$ and G-refinement of a subdivision of $[a, b]$ as in [1]. The concepts of singular graph, exceptional number, and summability set are as in [4]. If f and g are real-valued functions on an interval $[a, b]$ of the real axis, and if there is a summability set G for f and g in $[a, b]$, the Lane integral $L\int_a^b f dg$ is the refinement limit

$$\lim_{D\subset G, \supset} S_D(f, g),$$

and this integral is the Stieltjes mean sigma integral $Fm\int_a^b f dg$ in case the entire interval $[a, b]$ is a summability set for f and g in $[a, b]$.

In §2 we deal with a bounded real-valued function f on an interval $[a, b]$ of the real axis, a nondecreasing real-valued function g on the entire real axis and a subset G of $[a, b]$ dense on this interval and containing its endpoints. We suppose that if $H=(a, b)-G$ and H is nonempty then the ordered triple (g, f, H) is a singular graph, and that if y is in G and the restriction of g to $[a, b]$ is discontinuous at y, then y is not an exceptional number for f and g in $[a, b]$. We establish Theorem 2.1, which is analogous to Theorem 18 on page 278 of Graves [2] for the Stieltjes norm integral and the Stieltjes sigma integral, and which provides a set of necessary and sufficient conditions for G to be a summability set for f and g in $[a, b]$. Theorem 2.1 clearly provides as a special case a set of necessary and sufficient conditions for the Stieltjes mean sigma integral $Fm\int_a^b f dg$ to exist. We show how we have fairly simply from Theorem 2.1 the result due to Bzoch [1] that G is a summability set for f and g in $[a, b]$ if and only if there is a bounded

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real-valued function u on $[a, b]$ such that $u(x)=f(x)$ for all x in G and such that the Stieltjes mean sigma integral $Fm\int_a^b udg$ exists.

In §3 with f, g, G, and H as in §2 we use our Theorem 2.1 to obtain a necessary and sufficient condition for G to be a summability set for f and g in $[a, b]$ which is analogous to one given by C. B. Murray [5] for the Stieltjes mean sigma integral.

2. **An existence theorem.** Suppose in this section that f is a bounded real-valued function on an interval $[a, b]$ of the real axis, g is a nondecreasing real-valued function on the entire real axis, G is a subset of $[a, b]$ containing a and b which is dense on $[a, b]$, if $H=(a, b)-G$ and H is nonempty then (g, f, H) is a singular graph, and for each y in G if the restriction of g to $[a, b]$ is discontinuous at y then y is not an exceptional number for f and g in $[a, b]$. If $a \leqq y < b$ let $f_G(y^+)$ denote the limit

$$\lim_{x \to y^+, x \in G} f(x),$$

and similarly define $f_G(y^-)$ for $a < y \leqq b$. Let $M_G(f^+)$ denote the set of all numbers y such that $a \leqq y < b$ and $f_G(y^+)$ does not exist, and similarly define $M_G(f^-)$. Then let

$$M_G(f) = M_G(f^+) + M_G(f^-).$$

Also, let s and ψ be a nondecreasing saltus function on the entire real axis and a continuous nondecreasing function on the entire real axis, respectively, such that $g=s+\psi$. Let μ_ψ^* denote the outer measure function associated with the nondecreasing function ψ.

THEOREM 2.1. *For G to be a summability set for f and g in $[a, b]$, it is necessary and sufficient that the following statements hold*:

(a) *if $a \leqq y < b$ and $g(y^+) \neq g(y)$, then $f_G(y^+)$ exists*;

(b) *if $a < y \leqq b$ and $g(y^-) \neq g(y)$, then $f_G(y^-)$ exists*;

(c) $\mu_\psi^*(M_G(f))=0$.

PROOF OF THE SUFFICIENCY. We establish the sufficiency of the conjunction of (a), (b) and (c) by demonstrating that if these statements hold then

$$\lim_{D \subset G, \supset} S_D(f, s) \quad \text{and} \quad \lim_{D \subset G, \supset} S_D(f, \psi)$$

both exist, so that we can then conclude that

$$\lim_{D \subset G, \supset} S_D(f, g)$$

exists.

Proceeding in a manner analogous to that used by Lane in his proof of Theorem 3.1 of [3], we can show that the conjunction of (a) and (b) implies that the limit

$$\lim_{D\subset \tilde{G},\supset} S_D(f, s)$$

exists, where $\tilde{G}$ is the union of G and the set of points of (a, b) at which g is discontinuous. Then using the continuity of ψ in a modification of the sufficiency proof of Theorem 1 of [1] we can obtain the result that the conjunction of (a) and (b) implies that the limit

$$\lim_{D\subset G,\supset} S_D(f, s)$$

exists.

In this part of our sufficiency proof, we show that (c) implies that

$$\lim_{D\subset G,\supset} S_D(f, \psi)$$

exists. If A is a nonempty subset of G, let

$$\omega_G(A) = \underset{x \text{ in } A}{\text{l.u.b.}}\, f(x) - \underset{x \text{ in } A}{\text{g.l.b.}}\, f(x).$$

If y is in $[a, b]$, let

$$\omega_G(y) = \lim_{\delta\to 0^+} \omega_G((y-\delta, y+\delta)\cdot G).$$

For $\eta > 0$, let $R_G(\eta)$ be the set of all numbers y in (a, b) such that $\omega_G(y) \geqq \eta$, $y \notin G$, and $y \notin M_G(f)$. If $\eta > 0$, each point of $R_G(\eta)$ is an isolated point of this set, so that $R_G(\eta)$ is countable, and hence $\mu_\psi^*(R_G(\eta)) = 0$. For $\eta > 0$, let $T_G(\eta)$ denote the set of all y in $[a, b]$ such that $\omega_G(y) \geqq \eta$. If $\eta > 0$, $\mu_\psi^*(T_G(\eta)) = 0$, and $T_G(\eta)$ is closed. Proceeding as in the proof of the analogous result for the Stieltjes integral we can show that the desired limit exists.

PROOF OF THE NECESSITY. By assumption G is a summability set for f and g in $[a, b]$.

Suppose that (a) does not hold. Then there is a y in $[a, b)$ such that $g(y^+) \neq g(y)$ and $f_G(y^+)$ does not exist. Let $l = g(y^+) - g(y)$. There is a positive number k for which if $y < z \leqq b$ then there are points s and t in $G \cdot (y, z)$ such that $|f(s) - f(t)| \geqq k$. Let N be a positive number such that $|f(x)| \leqq N$ if $x \in [a, b]$. There is a number w in $(y, b]$ such that if $x \in (y, w)$ then $|g(x) - g(y^+)| < \min\{l/2, lk/16N\}$. Let $\overline{G} = G + \{y\}$. By Lemma 2 of [1], $\overline{G}$ is a summability set for f and g in $[a, b]$. Suppose that P is a $\overline{G}$-subdivision of $[a, b]$. Let $\overline{P}$ be a $\overline{G}$-refinement of P containing y and at least one point of (y, w). Let z be the smallest number in $\overline{P}\cdot(y, w)$. Let s and t be in $G\cdot(y, z)$ such that $|f(s)-f(t)| \geqq k$. Let $P'=\overline{P}+\{s\}$ and $P''=\overline{P}+\{t\}$. P' and P'' are $\overline{G}$-refinements of P, and

$$\begin{aligned} |S_{P'}(f, g) - S_{P''}(f, g)| &= \tfrac{1}{2}\{|f(s) - f(t)|\cdot|g(z) - g(y)| \\ &\qquad - |f(z) - f(y)|\cdot|g(s) - g(t)|\} \\ &> lk/8. \end{aligned}$$

This contradicts the result that $\overline{G}$ is a summability set for f and g in $[a, b]$.

We have from the above that (a) holds. Similarly (b) holds.

We now argue indirectly to show that (c) holds. Since by hypothesis

$$\lim_{D\subset G,\supset} S_D(f, g)$$

exists and since it follows from the result that (a) and (b) hold that

$$\lim_{D\subset G,\supset} S_D(f, s)$$

exists, we can conclude that

$$\lim_{D\subset G,\supset} S_D(f, \psi)$$

exists. Suppose that $\mu_\psi^*(M_G(f))>0$. Then at least one of $\mu_\psi^*(M_G(f^+))$ and $\mu_\psi^*(M_G(f^-))$ is positive; suppose for the sake of argument that $\mu_\psi^*(M_G(f^+))>0$. For each positive number δ, let $M_G(f, \delta^+)$ be the set of all x in $[a, b)$ such that

$$\left[\limsup_{t\to x^+, t \text{ in } G} f(t) - \liminf_{t\to x^+, t \text{ in } G} f(t)\right] \geqq \delta.$$

There is a positive integer j such that $\omega_j^+ \equiv \mu_\psi^*(M_G(f, (1/j)^+))>0$. Let $P=\{a=x_0<x_1< \cdots <x_n=b\}$ be a G-subdivision of $[a, b]$. Let $\{i_1<i_2< \cdots <i_m\}$ be the set of all positive integers i not exceeding n such that the segment (x_{i-1}, x_i) contains a point of $M_G(f, (1/j)^+)$. For $q=1, 2, \cdots, m$, let $u_q=$ l.u.b. $M_G(f, (1/j)^+)\cdot(x_{i_q-1}, x_{i_q})$. Using the continuity of ψ we can show that

$$\sum_{q=1}^{m} [\psi(u_q) - \psi(x_{i_q-1})] > \tfrac{1}{2}\omega_j^+.$$

For each $q=1, 2, \cdots, m$ for which $u_q=x_{i_q}$, or else $u_q<x_{i_q}$ and $u_q \notin M_G(f, (1/j)^+)$, let v_q be a number in (x_{i_q-1}, u_q) such that the inequality

$$(1) \qquad |\psi(u_q) - \psi(x)| < (1/4m)\cdot\omega_j^+$$

holds if $x\in(v_q, u_q)$, and such that the inequality

$$|\psi(x'') - \psi(x')| < (1/32mN)\cdot \overset{+}{\omega_j}\cdot 1/j \tag{2}$$

holds if x' and x'' are in (v_q, u_q). For each $q=1, 2, \cdots, m$ for which $u_q<x_{i_q}$ and $u_q\in M_G(f, (1/j)^+)$, let v_q be a number in (u_q, x_{i_q}) such that (1) holds if $x\in(u_q, v_q)$, and such that (2) holds if x' and x'' are in (u_q, v_q). For $q=1, 2, \cdots, m$, let s_q and t_q be numbers in G satisfying $v_q<s_q<t_q<u_q$ or $u_q<s_q<t_q<v_q$ according as $v_q<u_q$ or $u_q<v_q$, and such that $|f(t_q)-f(s_q)|>1/2j$. For $q=1, 2, \cdots, m$, let

$$\begin{aligned} d_q = {} & [f(x_{i_q-1}) + f(t_q)][\psi(t_q) - \psi(x_{i_q-1})] \\ & - \{[f(x_{i_q-1}) + f(s_q)][\psi(s_q) - \psi(x_{i_q-1})] \\ & \qquad + [f(s_q) + f(t_q)][\psi(t_q) - \psi(s_q)]\}. \end{aligned}$$

Let P_1 be the G-refinement of P such that $P_1=P+Q_1+\cdots+Q_m$ where for $q=1, 2, \cdots, m$, Q_q is $\{s_q, t_q\}$ or $\{t_q\}$ according as d_q is nonnegative or negative, and let P_2 be the G-refinement of P such that $P_2=P+R_1+\cdots+R_m$ where for $q=1, 2, \cdots, m$, R_q is $\{t_q\}$ or $\{s_q, t_q\}$ according as d_q is nonnegative or negative. We have that

$$\begin{aligned} |S_{P_1}(f, \psi) - S_{P_2}(f, \psi)| &= \tfrac{1}{2}\sum_{q=1}^{m} |d_q| \\ &\geqq \tfrac{1}{2}\sum_{q=1}^{m} \{|f(t_q) - f(s_q)|\,[\psi(s_q) - \psi(x_{i_q-1})] \\ &\qquad - |f(s_q) - f(x_{i_q-1})|\,[\psi(t_q) - \psi(s_q)]\} \\ &> \tfrac{1}{2}\{(1/2j)\cdot\tfrac{1}{4}\overset{+}{\omega_j} - (2Nm/32mN)\cdot\overset{+}{\omega_j}\cdot 1/j\} \\ &= (1/32)\overset{+}{\omega_j}\cdot 1/j. \end{aligned}$$

This contradicts the result that $\lim_{D\subset G}$, $S_D(f, \psi)$ exists. We conclude that $\mu_\psi^*(M_G(f))=0$.

We now show how the following result due to Bzoch [1] follows fairly simply from Theorem 2.1.

THEOREM 2.2. *For G to be a summability set for f and g in $[a, b]$, it is necessary and sufficient that there exists a bounded real-valued function u on $[a, b]$ such that $u(x)=f(x)$ for all x in G and such that the Stieltjes mean sigma integral $Fm\int_a^b udg$ exists.*

PROOF OF THE NECESSITY. It is easy to determine a bounded real-valued function u on $[a, b]$ such that

(i) $u(y)=f(y)$ if $y\in G$;

(ii) if $a \leqq y < b$ and $f_G(y^+)$ exists, then $u(y^+)$ exists;
(iii) if $a < y \leqq b$ and $f_G(y^-)$ exists, then $u(y^-)$ exists.

Let D be the set of points of discontinuity of u. Since $\mu_\psi^*(M_G(f))=0$ in view of Theorem 2.1, it follows that $\mu_\psi^*(D)=0$. It follows, then, from Theorem 2.1 that $Fm\int_a^b udg$ exists.

PROOF OF THE SUFFICIENCY. By assumption there is a bounded real-valued function u on $[a, b]$ such that $u(x)=f(x)$ for all x in G, and such that the Stieltjes mean sigma integral $Fm\int_a^b udg$ exists. Let D be the set of points of discontinuity of u. From Theorem 2.1 we have that $\mu_\psi^*(D)=0$. $M_G(f)\subset D$; hence $\mu_\psi^*(M_G(f))=0$. It follows, then, from Theorem 2.1 that the Lane integral $L\int_a^b fdg$ exists.

3. **A further existence theorem.** Let the functions f, g, s, ψ, ω_G and the sets G, H be as in §2. In the next theorem we present a necessary condition for G to be a summability set for f and g in $[a, b]$ which is analogous to one given by C. B. Murray [**5**] for the Stieltjes mean sigma integral. Using the hypothesis that (g, f, H) is a singular graph in a modification of the sufficiency proof of Theorem 1.2 of [**5**] we can show that the condition of the following theorem is also sufficient.

THEOREM 3.1. *Suppose that G is a summability set for f and g in $[a, b]$. Then for positive numbers ϵ and η there is a subdivision $D = \{a=y_0<y_1< \cdots <y_n=b\}$ of $[a, b]$ with the property that if $\{i_1<i_2< \cdots <i_p\}$ is the set of all positive integers i not exceeding n for which there are points x', x'' in $(y_{i-1}, y_i)\cdot G$ such that $|f(x')-f(x'')| \geqq \eta$, then $\sum_{j=1}^p[g(y_{i_j})-g(y_{i_j-1})]<\epsilon$.*

PROOF. We have from the sufficiency proof of Theorem 2.1 that there are a G-subdivision $\tilde{D}=\{a=t_0<t_1< \cdots <t_m=b\}$ of $[a, b]$ and a subset $\tilde{L}=\{k_1<k_2< \cdots <k_q\}$ of the set of all positive integers not exceeding m such that

$$\sum_{j=1}^{q} [\psi(t_{k_j}) - \psi(t_{k_j-1})] < \epsilon/2$$

and such that if k is a positive integer not exceeding m such that k is not in $\tilde{L}$ then $\omega_G([t_{k-1}, t_k]\cdot G)<\eta$.

Let j be a positive integer not exceeding q. Let $\{z_{i,j}\}_{i=1}^\infty$ be a sequence of distinct points of $[t_{k_j-1}, t_{k_j}]$ containing all of the points of this interval at which the saltus function s is discontinuous. For simplicity, let $z_{1,j}=t_{k_j-1}$ and $z_{2,j}=t_{k_j}$. Let $\bar{\imath}$ be a positive integer such that

$$\sum_{i=\bar{\imath}+2}^{\infty} [s(z_{i,j}^+) - s(z_{i,j}^-)] < \epsilon/2q.$$

Let $\Delta_j = \{t_{k_j-1} = u_{0,j} < u_{1,j} < \cdots < u_{\bar{i},j} = t_{k_j}\}$ be the subdivision of $[t_{k_j-1}, t_{k_j}]$ consisting of the numbers $z_{1,j}, z_{2,j}, \cdots, z_{\bar{i}+1,j}$. For $i = 1, 2, \cdots, \bar{i}$, let $u'_{i,j}$ be a number satisfying $u_{i-1,j} < u'_{i,j} < u_{i,j}$, and such that $\omega_G((u_{i-1,j}, u'_{i,j}) \cdot G) < \eta$ in case $g(u^+_{i-1,j}) \neq g(u_{i-1,j})$, and let $u''_{i,j}$ be a number satisfying $u'_{i,j} < u''_{i,j} < u_{i,j}$, and such that $\omega_G((u''_{i,j}, u_{i,j}) \cdot G) < \eta$ in case $g(u^-_{i,j}) \neq g(u_{i,j})$. Let $\tilde{D}_j = \{t_{k_j-1} = t_{0,j} < t_{1,j} < \cdots < t_{p_j,j} = t_{k_j}\}$ be the subdivision of $[t_{k_j-1}, t_{k_j}]$ consisting of the points of Δ_j, the numbers $u'_{1,j}, u'_{2,j}, \cdots, u'_{\bar{i},j}$, and the numbers $u''_{1,j}, u''_{2,j}, \cdots, u''_{\bar{i},j}$. Let $\{i_{1,j} < i_{2,j} < \cdots < i_{r_j,j}\}$ be the set of all positive integers i not exceeding p_j with $\omega_G((t_{i-1,j}, t_{i,j}) \cdot G) \geqq \eta$. We have that

$$\begin{aligned}\sum_{l=1}^{r_j} [s(t_{i_{l,j},j}) - s(t_{i_{l,j}-1,j})] &\leqq \sum_{i=1}^{\bar{i}} [s(u^-_{i,j}) - s(u^+_{i-1,j})] \\ &= \sum_{i=1}^{\bar{i}} \sum_{l: u_{i-1,j} < z_{l,j} < u_{i,j}} [s(z^+_{l,j}) - s(z^-_{l,j})] \\ &= \sum_{i=\bar{i}+2}^{\infty} [s(z^+_{i,j}) - s(z^-_{i,j})] < \epsilon/2q.\end{aligned}$$

Let $D = \{a = y_0 < y_1 < \cdots < y_n = b\}$ be the subdivision of $[a, b]$ equal to $\tilde{D} + \sum_{j=1}^{q} \tilde{D}_j$. Let $\{j_1 < j_2 < \cdots < j_r\}$ be the set of all positive integers j not exceeding n such that $\omega_G((y_{j-1}, y_j) \cdot G) \geqq \eta$. Then it follows that

$$\begin{aligned}\sum_{i=1}^{r} [g(y_{j_i}) - g(y_{j_i-1})] &\leqq \sum_{j=1}^{q} [\psi(t_{k_j}) - \psi(t_{k_j-1})] \\ &\quad + \sum_{j=1}^{q} \sum_{l=1}^{r_j} [s(t_{i_{l,j},j}) - s(t_{i_{l,j}-1,j})] < \epsilon.\end{aligned}$$

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