EXISTENCE CONDITIONS FOR THE LANE INTEGRAL

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1. Introduction. We deal in this paper with an integral which was defined by R. E. Lane [4] and which is an extension of the Stieltjes mean sigma integral introduced by H. L. Smith [6]. If $f$ and $g$ are real-valued functions on an interval $[a, b]$ of the real axis, and if $D = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ is a subdivision of $[a, b]$, we use the symbol $S_D(f, g)$ to denote the sum

$$\sum_{i=1}^n \frac{f(x_{i-1}) + f(x_i)}{2} \cdot [g(x_i) - g(x_{i-1})].$$

If $[a, b]$ is an interval of the real axis, and if $G$ is a subset of $[a, b]$ containing $a$ and $b$, we use the terminology $G$-subdivision of $[a, b]$ and $G$-refinement of a subdivision of $[a, b]$ as in [1]. The concepts of singular graph, exceptional number, and summability set are as in [4]. If $f$ and $g$ are real-valued functions on an interval $[a, b]$ of the real axis, and if there is a summability set $G$ for $f$ and $g$ in $[a, b]$, the Lane integral $L\int_a^b f dg$ is the refinement limit

$$\lim_{D \supseteq G} S_D(f, g),$$

and this integral is the Stieltjes mean sigma integral $Fm\int_a^b f dg$ in case the entire interval $[a, b]$ is a summability set for $f$ and $g$ in $[a, b]$.

In §2 we deal with a bounded real-valued function $f$ on an interval $[a, b]$ of the real axis, a nondecreasing real-valued function $g$ on the entire real axis and a subset $G$ of $[a, b]$ dense on this interval and containing its endpoints. We suppose that if $H = (a, b) - G$ and $H$ is nonempty then the ordered triple $(g, f, H)$ is a singular graph, and that if $y$ is in $G$ and the restriction of $g$ to $[a, b]$ is discontinuous at $y$, then $y$ is not an exceptional number for $f$ and $g$ in $[a, b]$. We establish Theorem 2.1, which is analogous to Theorem 18 on page 278 of Graves [2] for the Stieltjes norm integral and the Stieltjes sigma integral, and which provides a set of necessary and sufficient conditions for $G$ to be a summability set for $f$ and $g$ in $[a, b]$. Theorem 2.1 clearly provides as a special case a set of necessary and sufficient conditions for the Stieltjes mean sigma integral $Fm\int_a^b f dg$ to exist. We show how we have fairly simply from Theorem 2.1 the result due to Bzoch [1] that $G$ is a summability set for $f$ and $g$ in $[a, b]$ if and only if there is a bounded

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real-valued function \( u \) on \([a, b]\) such that \( u(x) = f(x) \) for all \( x \) in \( G \) and such that the Stieltjes mean sigma integral \( F_m \int_a^b u \, d\sigma \) exists.

In §3 with \( f, g, G, \) and \( H \) as in §2 we use our Theorem 2.1 to obtain a necessary and sufficient condition for \( G \) to be a summability set for \( f \) and \( g \) in \([a, b]\) which is analogous to one given by C. B. Murray [5] for the Stieltjes mean sigma integral.

2. **An existence theorem.** Suppose in this section that \( f \) is a bounded real-valued function on an interval \([a, b]\) of the real axis, \( g \) is a non-decreasing real-valued function on the entire real axis, \( G \) is a subset of \([a, b]\) containing \( a \) and \( b \) which is dense on \([a, b]\), if \( H = (a, b) - G \) and \( H \) is nonempty then \((g, f, H)\) is a singular graph, and for each \( y \) in \( G \) if the restriction of \( g \) to \([a, b]\) is discontinuous at \( y \) then \( y \) is not an exceptional number for \( f \) and \( g \) in \([a, b]\). If \( a \leq y < b \) let \( f_0(y+) \) denote the limit

\[
\lim_{x \to y^+, x \in G} f(x),
\]

and similarly define \( f_0(y-) \) for \( a < y \leq b \). Let \( M_0(f^+) \) denote the set of all numbers \( y \) such that \( a \leq y < b \) and \( f_0(y+) \) does not exist, and similarly define \( M_0(f^-) \). Then let

\[
M_0(f) = M_0(f^+) + M_0(f^-).
\]

Also, let \( s \) and \( \psi \) be a nondecreasing saltus function on the entire real axis and a continuous nondecreasing function on the entire real axis, respectively, such that \( g = s + \psi \). Let \( \mu_\psi^* \) denote the outer measure function associated with the nondecreasing function \( \psi \).

**Theorem 2.1.** For \( G \) to be a summability set for \( f \) and \( g \) in \([a, b]\), it is necessary and sufficient that the following statements hold:

(a) if \( a \leq y < b \) and \( g(y^+) \neq g(y) \), then \( f_0(y^+) \) exists;
(b) if \( a < y \leq b \) and \( g(y^-) \neq g(y) \), then \( f_0(y^-) \) exists;
(c) \( \mu_\psi^*(M_0(f)) = 0 \).

**Proof of the sufficiency.** We establish the sufficiency of the conjunction of (a), (b) and (c) by demonstrating that if these statements hold then

\[
\lim_{D \subseteq G, \sigma} S_D(f, s) \quad \text{and} \quad \lim_{D \subseteq G, \sigma} S_D(f, \psi)
\]

both exist, so that we can then conclude that

\[
\lim_{D \subseteq G, \sigma} S_D(f, g)
\]

exists.
Proceeding in a manner analogous to that used by Lane in his proof of Theorem 3.1 of [3], we can show that the conjunction of (a) and (b) implies that the limit
\[ \lim_{D \rightarrow \mathcal{G}, \tau} S_D(f, s) \]
exists, where \( \mathcal{G} \) is the union of \( G \) and the set of points of \((a, b)\) at which \( g \) is discontinuous. Then using the continuity of \( \psi \) in a modification of the sufficiency proof of Theorem 1 of [1] we can obtain the result that the conjunction of (a) and (b) implies that the limit
\[ \lim_{D \rightarrow \mathcal{G}, \tau} S_D(f, \psi) \]
exists.

In this part of our sufficiency proof, we show that (c) implies that
\[ \lim_{D \rightarrow \mathcal{G}, \tau} S_D(f, s) \]
exists. If \( A \) is a nonempty subset of \( G \), let
\[ \omega_G(A) = \text{l.u.b.}_{x \in A} f(x) - \text{g.l.b.}_{x \in A} f(x). \]
If \( y \) is in \([a, b]\), let
\[ \omega_G(y) = \lim_{\delta \rightarrow 0^+} \omega_G((y - \delta, y + \delta) \cap G). \]

For \( \eta > 0 \), let \( R_G(\eta) \) be the set of all numbers \( y \) in \((a, b)\) such that \( \omega_G(y) \geq \eta \), \( y \in G \), and \( y \in M_G(f) \). If \( \eta > 0 \), each point of \( R_G(\eta) \) is an isolated point of this set, so that \( R_G(\eta) \) is countable, and hence \( \mu_\psi^*(R_G(\eta)) = 0 \). For \( \eta > 0 \), let \( T_G(\eta) \) denote the set of all \( y \) in \([a, b]\) such that \( \omega_G(y) \geq \eta \). If \( \eta > 0 \), \( \mu_\psi^*(T_G(\eta)) = 0 \), and \( T_G(\eta) \) is closed. Proceeding as in the proof of the analogous result for the Stieltjes integral we can show that the desired limit exists.

**Proof of the necessity.** By assumption \( G \) is a summability set for \( f \) and \( g \) in \([a, b]\).

Suppose that (a) does not hold. Then there is a \( y \) in \([a, b]\) such that \( g(y^+) \neq g(y) \) and \( f_G(y^+) \) does not exist. Let \( l = g(y^+) - g(y) \). There is a positive number \( k \) for which if \( y < z \leq b \) then there are points \( s \) and \( t \) in \( G \cdot (y, z) \) such that \( |f(s) - f(t)| \geq k \). Let \( N \) be a positive number such that \( |f(x)| \leq N \) if \( x \in [a, b] \). There is a number \( w \) in \((y, w)\) such that if \( x \in (y, w) \) then \( |g(x) - g(y^+)| < \min \{l/2, lk/16N\} \). Let \( \mathcal{G} = G + \{y\} \).

By Lemma 2 of [1], \( \mathcal{G} \) is a summability set for \( f \) and \( g \) in \([a, b]\). Suppose that \( P \) is a \( \mathcal{G} \)-subdivision of \([a, b]\). Let \( \mathcal{P} \) be a \( \mathcal{G} \)-refinement of \( P \) containing \( y \) and at least one point of \((y, w)\). Let \( z \) be the smallest
number in $\bar{P}(y, w)$. Let $s$ and $t$ be in $G(y, z)$ such that $|f(s) - f(t)| \geq k$. Let $P' = \bar{P} + \{s\}$ and $P'' = \bar{P} + \{t\}$. $P'$ and $P''$ are $G$-refinements of $P$, and

$$\left| S_{P'}(f, g) - S_{P''}(f, g) \right| = \frac{1}{2}\left\{ |f(s) - f(t)| \cdot |g(z) - g(y)| - |f(z) - f(y)| \cdot |g(s) - g(t)| \right\} > \frac{l}{k8}.$$  

This contradicts the result that $G$ is a summability set for $f$ and $g$ in $[a, b]$.

We have from the above that (a) holds. Similarly (b) holds.

We now argue indirectly to show that (c) holds. Since by hypothesis

$$\lim_{D \to G, \nu} S_D(f, g)$$

exists and since it follows from the result that (a) and (b) hold that

$$\lim_{D \to G, \nu} S_D(f, s)$$

exists, we can conclude that

$$\lim_{D \to G, \nu} S_D(f, \psi)$$

exists. Suppose that $\mu_\phi^+(M_G(f)) > 0$. Then at least one of $\mu_\phi^+(M_G(f^+))$ and $\mu_\phi^+(M_G(f^-))$ is positive; suppose for the sake of argument that $\mu_\phi^+(M_G(f^+)) > 0$. For each positive number $\delta$, let $M_G(f, \delta^+)$ be the set of all $x$ in $[a, b]$ such that

$$\lim_{t \to x^+, t \in G} f(t) - \liminf_{t \to x^+, t \in G} f(t) \geq \delta.$$  

There is a positive integer $j$ such that $\omega_j^+ \equiv \mu_\phi^+(M_G(f, (1/j)^+)) > 0$. Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a $G$-subdivision of $[a, b]$. Let $\{i_1 < i_2 < \cdots < i_m\}$ be the set of all positive integers $i$ not exceeding $n$ such that the segment $(x_{i-1}, x_i)$ contains a point of $M_G(f, (1/j)^+)$. For each integer $q = 1, 2, \cdots, m$, let $u_q = \text{l.u.b.} M_G(f, (1/j)^+) \cdot (x_{i_q-1}, x_{i_q})$. Using the continuity of $\psi$ we can show that

$$\sum_{q=1}^m [\psi(u_q) - \psi(x_{i_q-1})] > \frac{1}{2}\omega_j^+.$$  

For each $q = 1, 2, \cdots, m$ for which $u_q = x_{i_q}$, or else $u_q < x_{i_q}$ and $u_q \in M_G(f, (1/j)^+)$, let $v_q$ be a number in $(x_{i_q-1}, u_q)$ such that the inequality

$$|\psi(u_q) - \psi(x)| < (1/4m) \cdot \omega_j^+$$
holds if \( x \in (v_q, u_q) \), and such that the inequality

\[
(2) \quad |\psi(x''') - \psi(x')| < (1/32mN) \cdot \omega^+ \cdot 1/j
\]

holds if \( x' \) and \( x''' \) are in \((v_q, u_q)\). For each \( q = 1, 2, \ldots, m \) for which \( u_q < x_{iq} \) and \( u_q \in M_\alpha(f, (1/j)^+) \), let \( v_q \) be a number in \((u_q, x_{iq})\) such that

\( (1) \) holds if \( x \in (u_q, v_q) \), and such that \( (2) \) holds if \( x' \) and \( x'' \) are in 
\((u_q, v_q)\). For \( q = 1, 2, \ldots, m \), let \( s_q \) and \( t_q \) be numbers in \( G \) satisfying 
\( v_q < s_q < t_q < u_q \) or \( u_q < s_q < t_q < v_q \) according as \( v_q < u_q \) or \( u_q < v_q \), and such that

\[
|f(t_q) - f(s_q)| > 1/2j.
\]

For \( q = 1, 2, \ldots, m \), let

\[
d_q = |f(x_{iq-1}) + f(t_q)| |\psi(t_q) - \psi(x_{iq-1})| - |f(x_{iq-1}) + f(s_q)| |\psi(s_q) - \psi(x_{iq-1})|
\]

\[
+ |f(s_q) + f(t_q)| |\psi(t_q) - \psi(s_q)|.
\]

Let \( P_1 \) be the \( G \)-refinement of \( P \) such that \( P_1 = P + Q_1 + \cdots + Q_m \)
where for \( q = 1, 2, \ldots, m \), \( Q_q \) is \( \{s_q, t_q\} \) or \( \{t_q\} \) according as \( d_q \) is
nonnegative or negative, and let \( P_2 \) be the \( G \)-refinement of \( P \) such
that \( P_2 = P + R_1 + \cdots + R_m \) where for \( q = 1, 2, \ldots, m \), \( R_q \) is \( \{t_q\} \)
or \( \{s_q, t_q\} \) according as \( d_q \) is nonnegative or negative. We have that

\[
|S_{P_1}(f, \psi) - S_{P_2}(f, \psi)| = \frac{1}{2} \sum_{q=1}^{m} |d_q|
\]

\[
\geq \frac{1}{2} \sum_{q=1}^{m} \left| f(t_q) - f(s_q) \right| \left| \psi(s_q) - \psi(x_{iq-1}) \right| - \left| f(s_q) - f(x_{iq-1}) \right| \left| \psi(t_q) - \psi(s_q) \right|
\]

\[
> \frac{1}{2} \left\{ (1/2j) \cdot \omega^+ - (2Nm/32mN) \cdot \omega^+ \cdot 1/j \right\}
\]

\[
= (1/32) \omega^+ \cdot 1/j.
\]

This contradicts the result that \( \lim_{D \to G} S_D(f, \psi) \) exists. We conclude
that \( \mu^{\psi}_{\alpha}(M_\alpha(f)) = 0 \).

We now show how the following result due to Bzoch [1] follows
fairly simply from Theorem 2.1.

**Theorem 2.2.** For \( G \) to be a summability set for \( f \) and \( g \) in \([a, b]\), it
is necessary and sufficient that there exists a bounded real-valued function
\( u \) on \([a, b]\) such that \( u(x) = f(x) \) for all \( x \) in \( G \) and such that the Stieltjes
mean sigma integral \( Fm^\alpha_{\psi} udg \) exists.

**Proof of the necessity.** It is easy to determine a bounded real-valued function \( u \) on \([a, b]\) such that

(i) \( u(y) = f(y) \) if \( y \in G \);
(ii) if $a \leq y < b$ and $f_{\varnothing}(y^+)$ exists, then $u(y^+)$ exists;
(iii) if $a < y \leq b$ and $f_{\varnothing}(y^-)$ exists, then $u(y^-)$ exists.

Let $D$ be the set of points of discontinuity of $u$. Since $\mu_{\varnothing}^*(M_{\varnothing}(f)) = 0$ in view of Theorem 2.1, it follows that $\mu_{\varnothing}^*(D) = 0$. It follows, then, from Theorem 2.1 that $Fm\int_0^a u dg$ exists.

Proof of the sufficiency. By assumption there is a bounded real-valued function $u$ on $[a, b]$ such that $u(x) = f(x)$ for all $x$ in $G$, and such that the Stieltjes mean sigma integral $Fm\int_0^a u dg$ exists. Let $D$ be the set of points of discontinuity of $u$. From Theorem 2.1 we have that $\mu_{\varnothing}^*(D) = 0$. $M_{\varnothing}(f) \subset D$; hence $\mu_{\varnothing}^*(M_{\varnothing}(f)) = 0$. It follows, then, from Theorem 2.1 that the Lane integral $L\int_0^a f dg$ exists.

3. A further existence theorem. Let the functions $f$, $g$, $s$, $\psi$, $\omega_G$ and the sets $G$, $H$ be as in §2. In the next theorem we present a necessary condition for $G$ to be a summability set for $f$ and $g$ in $[a, b]$ which is analogous to one given by C. B. Murray [5] for the Stieltjes mean sigma integral. Using the hypothesis that $(g, f, H)$ is a singular graph in a modification of the sufficiency proof of Theorem 1.2 of [5] we can show that the condition of the following theorem is also sufficient.

Theorem 3.1. Suppose that $G$ is a summability set for $f$ and $g$ in $[a, b]$. Then for positive numbers $\epsilon$ and $\eta$ there is a subdivision $D = \{a = y_0 < y_1 < \cdots < y_n = b\}$ of $[a, b]$ with the property that if $i_1 < i_2 < \cdots < i_p$ is the set of all positive integers $i$ not exceeding $n$ for which there are points $x', x''$ in $(y_{i-1}, y_i) \cdot G$ such that $|f(x') - f(x'')| \geq \eta$, then $\sum_{i=1}^{p} |g(y_{i_i}) - g(y_{j_i})| < \epsilon$.

Proof. We have from the sufficiency proof of Theorem 2.1 that there are a $G$-subdivision $\tilde{D} = \{a = t_0 < t_1 < \cdots < t_m = b\}$ of $[a, b]$ and a subset $\tilde{L} = \{k_1 < k_2 < \cdots < k_q\}$ of the set of all positive integers not exceeding $m$ such that

$$\sum_{j=1}^{q} [\psi(t_{k_j}) - \psi(t_{k_{j-1}})] < \frac{\epsilon}{2}$$

and such that if $k$ is a positive integer not exceeding $m$ such that $k$ is not in $\tilde{L}$ then $\omega_G([t_{k-1}, t_k) \cdot G) < \eta$.

Let $j$ be a positive integer not exceeding $q$. Let $\{z_{i,j}\}_{i=1}^{q}$ be a sequence of distinct points of $[t_{k_{j-1}}, t_{k_j}]$ containing all of the points of this interval at which the saltus function $s$ is discontinuous. For simplicity, let $z_{1,j} = t_{k_{j-1}}$ and $z_{2,j} = t_{k_j}$. Let $i$ be a positive integer such that

$$\sum_{i=1}^{\infty} [s(z_{i,j}^+) - s(z_{i,j}^-)] < \frac{\epsilon}{2q}.$$
Let $\Delta_j = \{ t_{k_j-1} = u_{0,j} < u_{1,j} < \cdots < u_{i,j} = t_{k_j} \}$ be the subdivision of $[t_{k_j-1}, t_{k_j}]$ consisting of the numbers $z_{1,j}, z_{2,j}, \ldots, z_{i+1,j}$. For $i = 1, 2, \ldots, q,$ let $u_{i,j}'$ be a number satisfying $u_{i-1,j} < u_{i,j}' < u_{i,j},$ and such that $\omega_\alpha((u_{i-1,j}, u_{i,j}') \cdot G) < \eta$ in case $g(u_{i-1,j}) \neq g(u_{i,j}),$ and let $u_{i,j}''$ be a number satisfying $u_{i,j}' < u_{i,j}'' < u_{i,j},$ and such that $\omega_\alpha((u_{i,j}'', u_{i,j}) \cdot G) < \eta$ in case $g(u_{i,j}') \neq g(u_{i,j}).$ Let $\tilde{D}_j = \{ t_{k_j-1} = t_{0,j} < t_{1,j} < \cdots < t_{p_j,j} = t_{k_j} \}$ be the subdivision of $[t_{k_j-1}, t_{k_j}]$ consisting of the points of $\Delta_j,$ the numbers $u_{1,j}', u_{2,j}', \ldots, u_{i,j}',$ and the numbers $u_{1,j}'', u_{2,j}'', \ldots, u_{i,j}''.$ Let $\{ t_{i,j} < t_{2,j} < \cdots < t_{r_j,j} \}$ be the set of all positive integers $i$ not exceeding $p_j$ with $\omega_\alpha((t_{i,j-1}, t_{i,j}) \cdot G) \geq \eta.$ We have that

$$\sum_{i=1}^{q} \sum_{l=1}^{r_j} [s(t_{i,j}) - s(t_{i,j-1})] \leq \sum_{i=1}^{q} \sum_{l=1}^{r_j} [s(u_{i,j}^+) - s(u_{i,j}^-)]$$

$$= \sum_{i=1}^{q} \sum_{l: u_{i,j-1} < z_{i,j} < u_{i,j}} [s(z_{i,j}^+) - s(z_{i,j}^-)]$$

$$= \sum_{i=1}^{q} \sum_{l: z_{i,j} < z_{i,j} < u_{i,j}} [s(z_{i,j}^+) - s(z_{i,j}^-)] < \epsilon/2q.$$

Let $D = \{ a = y_0 < y_1 < \cdots < y_n = b \}$ be the subdivision of $[a, b]$ equal to $\tilde{D} + \sum_{j=1}^{r} \tilde{D}_j.$ Let $\{ j_1 < j_2 < \cdots < j_r \}$ be the set of all positive integers $j$ not exceeding $n$ such that $\omega_\alpha((y_{j-1}, y_j) \cdot G) \geq \eta.$ Then it follows that

$$\sum_{i=1}^{r} [g(y_{j_i}) - g(y_{j_i-1})] \leq \sum_{j=1}^{q} [\Psi(t_{k_j}) - \Psi(t_{k_j-1})]$$

$$+ \sum_{j=1}^{q} \sum_{l=1}^{r_j} [s(t_{i,j}) - s(t_{i,j-1})] < \epsilon.$$

### References


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