METRIZATION OF PROXIMITY SPACES[1]

SOLOMON LEADER

0. **Introduction.** Our purpose here is to present a metrization criterion for proximity spaces that is analogous to R. L. Moore's criterion for topological spaces [8]. Where Moore's criterion demands a sequence of open coverings, our criterion requires "admissible" coverings. In Moore's criterion a set A is close to a point b (that is, $b \in \overline{A}$) if and only if the star of A meets the star of b for every covering in the sequence. For proximity spaces we require only that a set A be close to a set B if and only if the star of A meets B for every covering in the sequence. Such weakening of the star-separation condition is possible because proximity spaces have a strong separation axiom. However, we must require that our sequence of coverings be nested by refinement.

Our covering criterion is easily translated into a criterion involving "admissible" entourages. These entourages play a key role in the criterion of Efremovič and Švarc [6]. Their criterion easily implies ours, but the converse is much more difficult. It should therefore be easier to prove a given proximity space metrizable using our criterion rather than that of [6].

1. **The Metrization Theorem.** Let (X, β) be a proximity space in the sense of Efremovič [3]–[5]. We recall some basic definitions. B is a *β-neighborhood* of A whenever $A \not\beta X-B$. (X, β) is *metrizable* if there exists a metric σ on X such that for all subsets A, B of X

(1) $A \beta B$ if and only if $\sigma(A, B) = \inf\{\sigma(a, b): (a, b) \in A \times B\} = 0.$

A pseudometric σ on X is a *β-gauge* [10] if the direct implication holds in (1).

We shall call a subset P of the cartesian product X^2 *compressed* if P is infinite and $A \beta B$ for every pair of subsets A, B of X with $(A \times B) \cap P$ infinite. A covering U of X is *admissible* if for every compressed set P there exist (x, y) in P and E in U such that both x and y belong to E. An entourage (subset of X^2 containing the diagonal I) U is *admissible* if U meets every compressed subset P of X^2. Since every infinite subset of a compressed set is compressed, U must contain almost all (all but finitely many points) of P.

We can now state the Metrization Theorem. Its proof will be given in §5.

Received by the editors September 23, 1966.

[1] Work done under NSF GP-4413.

THEOREM I. *For any proximity space* (X, β) *the following three conditions are equivalent*:

(i) (X, β) *is metrizable.*

(ii) *There exists a sequence* $\{\mathbf{U}_n\}$ *of admissible coverings with* $\mathbf{U}_{n+1}$ *a refinement of* $\mathbf{U}_n$ *and*

(2) $A \beta B$ *if and only if for every* n *some member of* $\mathbf{U}_n$ *meets both* A *and* B.

(iii) *There exists a sequence* $\{U_n\}$ *of admissible symmetric entourages with* $U_{n+1} \subseteq U_n$ *and*

$$A \beta B \text{ if and only if } A \times B \text{ meets } U_n \text{ for all } n. \tag{3}$$

2. **Equivalent sequences.** An entourage U is a *β-entourage* if $U[A]$ is a β-neighborhood of A for every subset A of X. An entourage U is *simple* if it is the complement of a product:

$$U = X^2 - A \times B. \tag{4}$$

Clearly, the entourage (4) is a β-entourage if and only if $A \not\beta B$.

We can now characterize the equivalence relation $\{x_n\} \sim \{y_n\}$ introduced in [6].

THEOREM II. *For any sequence* $\{(x_n, y_n)\}$ *in* X^2 *the following seven conditions are equivalent*:

(i) *For every infinite set* M *of positive integers* $\{x_n\colon n \in M\}$ β $\{y_n\colon n \in M\}$.

(ii) *If* $(x_n, y_n) \in A \times B$ *for infinitely many* n, *then* $A \beta B$.

(iii) $\operatorname{Lim}_{n\to\infty} \sigma(x_n, y_n) = 0$ *for every β-gauge* σ.

(iv) $\operatorname{Lim}_{n\to\infty} \sigma(x_n, y_n) = 0$ *for every totally bounded β-gauge* σ.

(v) *Given any simple β-entourage* U, $(x_n, y_n) \in U$ *for almost all* n.

(vi) *Given any entourage* U *in the precompact uniform structure associated* [1], [7] *with* β, $(x_n, y_n) \in U$ *for almost all* n.

(vii) *Given any entourage* U *in the total structure associated* [2] *with* β, $(x_n, y_n) \in U$ *for almost all* n.

PROOF. The equivalence of (i) and (ii) is trivial. The equivalence of (ii) and (iii) is just part (b) of Lemma 3 in [10]. (iii) is equivalent to (viii) because the total β-structure is generated by the β-gauges [2], [11]. Analogously, (iv) is equivalent to (vi) because the precompact β-structure is generated by the totally bounded β-gauges [11]. (v) is equivalent to (vi) because the simple β-entourages form a subbase for the precompact β-structure [1], [7]. (iii) implies (iv) a fortiori. Finally, (iv) implies (ii) because $A \beta B$ if and only if $\sigma(A, B) = 0$ for every totally bounded β-gauge [5], [10].

3. Compressed sets.

THEOREM III. *For any infinite subset P of X^2 the following eight conditions are equivalent*:

(o) *P is compressed.*

(i) *If Q is any infinite subset of P, then for $\pi_j(x_1, x_2)=x_j$ with $j=1, 2$, $\pi_1 Q \,\beta\, \pi_2 Q$.*

(ii) *If $\{(x_n, y_n)\}$ is any sequence of distinct points in P, then $\{x_n\} \sim \{y_n\}$.*

(iii) *Given any β-gauge σ and $\epsilon > 0$,*

$$\sigma(x, y) \leqq \epsilon \text{ for almost all } (x, y) \text{ in } P. \tag{5}$$

(iv) *Given any totally bounded β-gauge σ and $\epsilon > 0$, (5) holds.*

(v) *Every simple β-entourage (4) contains almost all of P.*

(vi) *Every entourage in the precompact β-structure contains almost all of P.*

(vii) *Every entourage in the total β-structure contains almost all of P.*

PROOF. Defining $A=\pi_1 Q$ and $B=\pi_2 Q$ we have $Q\subseteq(A\times B)\cap P$. Hence (o) implies (i). To prove the converse apply (i) with $Q=(A\times B)\cap P$ to get $\pi_1 Q\subseteq A$ and $\pi_2 Q\subseteq B$. Then, for Q infinite, (i) implies $A \,\beta\, B$, hence (o).

To prove (i) implies (ii) apply (i) with $Q=\{(x_n, y_n): n\in M\}$ to get (i) of Theorem II. The converse follows similarly by applying (ii) to a sequence of distinct points in Q.

The equivalence of (ii) with each of the conditions (iii)—(vii) follows from Theorem II using the respective characterizations (iii)—(vii) of sequential equivalence.

4. Admissible entourages.

We next show that the admissible entourages are precisely those introduced in [6].

THEOREM IV. *An entourage U is admissible if and only if for every pair of equivalent sequences $\{x_n\}\sim\{y_n\}$ in X, $(x_n, y_n)\in U$ for some (hence, almost all) n.*

PROOF. Let U be admissible and $\{x_n\}\sim\{y_n\}$. Let P be the range of the double sequence $\{(x_n, y_n)\}$. We contend U meets P.

If P is finite, then the double sequence has a constant subsequence. By (i) of Theorem II a constant subsequence must lie in the diagonal I since X is Hausdorff. Since U contains I, U meets P.

On the other hand, if P is infinite then, according to (i) of Theorems II and III, P is compressed. Hence, U meets P.

Conversely, let an entourage U satisfy the sequential condition. Given any compressed set P, choose a sequence $\{(x_n, y_n)\}$ of distinct points in P to conclude from (ii) of Theorem III that $\{x_n\} \sim \{y_n\}$. Hence $(x_n, y_n) \in U$ for some n. So U meets P and is therefore admissible.

5. **Proof of Theorem I.** Recall that every covering $\boldsymbol{U}$ induces a symmetric entourage U, namely

$$U = \bigcup \{E^2 \colon E \in \boldsymbol{U}\}. \tag{6}$$

Comparison of our definitions immediately shows that under (6) U is an admissible entourage if and only if $\boldsymbol{U}$ is an admissible covering.

Given (i) of Theorem I let σ be a metric for (X, β). Let $\boldsymbol{U}_n$ be the covering consisting of all subsets of X with σ-diameter at most $1/n$. Then (2) follows immediately from (1). Moreover, $\boldsymbol{U}_{n+1}$ is a refinement of $\boldsymbol{U}_n$ since it is actually a subcovering. To show that each $\boldsymbol{U}_n$ is an admissible covering we need only note that the entourage $U_n = \sigma^{-1}[0, 1/n]$ induced by (6) is admissible because (o) implies (iii) in Theorem III. Hence (i) implies (ii).

To prove (ii) implies (iii) let U_n be the admissible symmetric entourage (6) induced by the admissible covering $\boldsymbol{U}_n$. Then (2) translates via (6) into (3). Moreover, since $\boldsymbol{U}_{n+1}$ refines $\boldsymbol{U}_n$, $U_{n+1} \subseteq U_n$.

To prove (iii) implies (i) it suffices to prove, in view of (1), (3), and the Metrization Lemma [9], that $\{U_n\}$ is a base for a uniform structure. That is, we must show that for each m there exists n with $U_n^2 \subseteq U_m$. Suppose this were false. Then there would exist m such that for all n we could choose (x_n, y_n) in $U_n^2 - U_m$. We could thereby also choose z_n for all n such that both (x_n, z_n) and (z_n, y_n) belong to U_n, while

$$(x_n, y_n) \notin U_m. \tag{7}$$

Applying (3) to get (ii) of Theorem II, we conclude that $\{x_n\} \sim \{z_n\}$ and $\{z_n\} \sim \{y_n\}$. Hence, by (iii) of Theorem II and the triangle inequality, $\{x_n\} \sim \{y_n\}$ which according to Theorem IV contradicts (7).

References

1. E. M. Alfsen and J. E. Fenstad, *On the equivalence between proximity structures and totally bounded uniform structures*, Math. Scand. **7** (1959), 353–360.

2. E. M. Alfsen and O. Njåstad, *Proximity and generalized uniformity*, Fund. Math. **52** (1963), 235–252.

3. V. A. Efremovič, *Infinitesimal spaces*, Dokl. Akad. Nauk SSSR **76** (1951), 341–343.

4. ——— *Infinitesimal spaces*, Uspehi Mat. Nauk **6** (1951), no. 4 (44), 203–204.

5. ——— *The geometry of proximity*, Mat. Sb. **31** (**73**) (1952), 189–200.

6. V. A. Efremovič and A. S. Švarc, *A new definition of uniform spaces. Metrization of proximity spaces.* Dokl. Akad. Nauk SSSR **89** (1953), 393–396.

7. I. S. Gál, *Proximity relations and precompact structures*, Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 304–326.

8. F. B. Jones, *Metrization*, Amer. Math. Monthly (73) **6** (1966), 571–576.

9. J. L. Kelley, *General topology*, Van Nostrand, Princeton, N. J., 1955.

10. S. Leader, *On completion of proximity spaces by local clusters*, Fund. Math. **48** (1960), 201–216.

11. ——— *On pseudometrics for generalized uniform structures*, Proc. Amer. Math. Soc. **16** (1965), 493–495.

RUTGERS—THE STATE UNIVERSITY