COMPACTNESS OF MAPPINGS ON PRODUCTS OF LOCALLY CONNECTED GENERALIZED CONTINUA

R. F. DICKMAN, JR.

DEFINITION. A connected topological space has the *complementation property* provided the complement of any compact set has at most one nonconditionally compact component.

Throughout this paper let X and Y denote noncompact locally connected generalized continua and let R denote a regular curve, i.e. let R denote a locally connected generalized continuum that has the property that for any point p in R and any open set U of R containing p there exists an open set V of R containing p such that $V\subset U$ and Fr V is a finite set. By a mapping we will always mean a continuous function.

THEOREM. *The product space $Z=X\times Y$ has the complementation property.*

PROOF. Let K be a compact set in Z and let U and V be conditionally compact open subsets of X and Y respectively such that $K\subset(U\times V)$. Let P be any nonempty component of $X-\overline{U}$ and let Q be any nonempty component of $X-\overline{V}$. Then the connected set $(P\times Y)+(X\times Q)$ intersects every component of $H=Z-(\overline{U}\times\overline{V})$ and hence H is connected. This implies that $Z-K$ has exactly one nonconditionally compact component.

COROLLARY 1. *Let f be a mapping of $Z=X\times Y$ onto a Hausdorff space W and suppose that point inverses of f have compact boundaries. Then if some point inverse $A=f^{-1}(p)$ is not compact, $H=Z-\operatorname{int} A$ is a compact set that maps onto W and hence f is a closed mapping.*

PROOF. Since $F=\operatorname{Fr} A$ is compact and Z is locally connected, there is a nonconditionally compact component Q of $Z-F$ that lies entirely in int A. By the theorem above Q is the only nonconditionally compact component of $Z-F$ and thus $Z-Q$ is a compact set. Since H is a closed subset of $Z-Q$, H is compact.

COROLLARY 2. *Let f be a closed mapping of $Z=X\times Y$ onto a noncompact metric space W. Then f is a compact mapping.*

PROOF. By a well-known result of Vaĭnšteĭn, point inverses of f have compact boundaries; thus by Corollary (1), point inverses of f

Received by the editors September 29, 1966.

are compact. Closed mappings with compact point inverses are known to be compact mappings.

COROLLARY 3. *Let f be a mapping of $Z=X\times Y$ into the regular curve R and suppose that point inverses of f have compact boundaries. Then* (1) *if the closure of $f(Z)$ in R is not compact, f is a compact mapping*; *and* (2) *if the closure of $f(Z)$ in R is compact, then for any compactification C of Z there is a continuous extension of f to all of C.*

PROOF OF (1). Suppose that f is not compact. Then there exists a sequence $\{x_i\}$ in Z such that $\{x_i\}$ does not have any convergent subsequences and such that $\{f(x_i)\}$ converges to some point y in R. Let V be any conditionally compact open set containing y such that $F=\mathrm{Fr}\ V$ is a finite set. By Corollary (1) $K=f^{-1}(F)$ is a compact set. Furthermore since $P=f^{-1}(V)$ is not conditionally compact, P contains a nonconditionally compact component of $Z-K$ and since the closure of $f(Z)$ is not compact, $Q=f^{-1}(R-\overline{V})$ is not conditionally compact and hence also contains a nonconditionally compact component of $Z-K$. But this is a contradiction since Z has the complementation property. Hence f is a compact mapping.

Part (2) is a consequence of Theorem (3.1) of [1].

COROLLARY 3.1. *Let f be a real-valued mapping defined on $X\times Y$ such that boundaries of point inverses are compact. Then* (1) *if f is not bounded, f is a compact mapping*; *and* (2) *if f is bounded, then for any compactification C of $X\times Y$ there is a continuous extension of f to all of C.*

COROLLARY 4. *Let f be any reflexive compact mapping of $Z=X\times Y$ onto E^2, i.e. let f be any mapping of Z onto E^2 such that for any compact set A in Z, $f^{-1}f(A)$ is also compact. Then f is a compact mapping.*

PROOF. The corollary is a consequence of the above theorem and Theorem 6 of [2].

COROLLARY 4.1. *Let f be any* 1-1 *mapping of $Z=X\times Y$ onto E^2. Then f is a homeomorphism.*

REFERENCES

1. R. F. Dickman, Jr., *Unicoherence and related properties*, Duke Math. J. **31** (1967), 343–352.

2. E. Duda, *Reflexive compact mappings*, Proc. Amer. Math. Soc. **17** (1966), 688–693.

UNIVERSITY OF MIAMI