EXISTENCE, UNIQUENESS, AND SPECTRAL PROPERTIES OF NONLINEAR EIGENVALUE PROBLEMS

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We consider the following nonlinear eigenvalue problem:

\begin{align*}
(1) \quad (p(x)u')' + \lambda f(x, u) &= 0, \quad 0 \leq x \leq 1, \\
(2) \quad a_0u(0) - a_1u'(0) &= 0, \quad |a_0| + |a_1| \neq 0, \\
(3) \quad b_0u(1) + b_1u'(1) &= 0, \quad |b_0| + |b_1| \neq 0.
\end{align*}

We suppose that \( p(x) > 0 \) and \( p'(x) \) is continuous on \( 0 \leq x \leq 1 \) and that \( f(x, u) \) satisfies the following conditions:

H-1: \( f(x, u) \) is continuously differentiable in \( D \):
\[ 0 \leq x \leq 1, \quad -\infty < u < \infty. \]
H-2: \( 0 < f_u(x, u) < \rho(x) \) on \( D \), where \( \rho(x) > 0 \) in \( 0 \leq x \leq 1 \).
H-3: \( f(x, 0) \neq 0 \) on \( 0 \leq x \leq 1 \).

Our main result is the

**Theorem.** Let \( f(x, u) \) satisfy H-1, 2, 3, and let the constants \( a_i, b_i \) satisfy
\[ a_i \geq 0, \quad b_i \geq 0, \quad (i = 0, 1), \quad a_0 + b_0 > 0. \]

Then, there exists a unique solution of (1), (2), (3) for all \( \lambda \) in \( 0 < \lambda < \mu_1\{\rho\} \), where \( \mu_1\{\rho\} \) is the principal (i.e., least) eigenvalue of

\begin{align*}
(4) \quad (p(x)u')' + \mu p(x)u &= 0, \quad 0 \leq x \leq 1, \\
(5) \quad a_0u(0) - a_1u'(0) &= 0, \\
(6) \quad b_0u(1) + b_1u'(1) &= 0.
\end{align*}

**Proof.** We outline the proof which is based on a technique used recently by H. B. Keller [1]. The initial value problem

\begin{align*}
(\rho(x)y')' + \lambda f(x, y) &= 0, \\
\alpha_0y(0) - \alpha_1y'(0) &= 0, \\
\gamma_0y(0) - \gamma_1y'(0) &= s, \quad \alpha_1\gamma_0 - \alpha_0\gamma_1 = 1,
\end{align*}

has the unique solution \( y(s; x) \). The problem (1), (2), (3) has as many solutions as there are real roots, \( s^* \), of
\[ \phi(s) \equiv b_0 y(s; 1) + b_1 y'(s; 1) = 0. \]

We shall show that \( \phi'(s) \) is positive and bounded away from zero, from which it follows that \( \phi(s) = 0 \) always has one and only one root.

Since \( y(s; x) \) is continuously differentiable with respect to \( s \), the derivative \( \psi(x) \equiv \partial y(s; x)/\partial s \) satisfies the variational problem

\[
\begin{align*}
\tag{7} (\phi(x)\psi')' + \lambda f_u(x, y)\psi &= 0, \\
\tag{8} a_0 \psi(0) - a_1 \psi'(0) &= 0, \\
\tag{9} c_0 \psi(0) - c_1 \psi'(0) &= 1.
\end{align*}
\]

Clearly we must show that \( \phi'(s) = b_0 \psi(1) + b_1 \psi'(1) \) is positive and bounded away from zero. To do this we consider the linear problem

\[
\begin{align*}
\tag{10} (\phi(x)v')' + \lambda \rho(x)v &= 0, \\
\tag{11} a_0 v(0) - a_1 v'(0) &= 0, \\
\tag{12} c_0 v(0) - c_1 v'(0) &= 1.
\end{align*}
\]

For a fixed \( \lambda = \lambda_1 \), say, let \( l \) be the first value of \( x > 0 \) at which \( b_0 \psi(l) + b_1 \psi'(l) = 0 \). (That such an \( l \) exists will be clear from the formulation of problem (13), (14), (15).) Then, the unique solution \( v_1(x) \) of (10), (11), (12) also satisfies

\[
\begin{align*}
\tag{13} (\phi(x)v_1')' + \lambda_1 \rho(x)v_1 &= 0, \\
\tag{14} a_0 v_1(0) - a_1 v_1'(0) &= 0, \\
\tag{15} b_0 v_1(l) + b_1 v_1'(l) &= 0.
\end{align*}
\]

where \( \lambda_1 = \lambda_1(l) \) is the principal eigenvalue of (13), (14), (15) and \( v_1(x) \) is the corresponding eigenfunction normalized so that it satisfies (12).

We now show that \( b_0 \psi(x) + b_1 \psi'(x) > 0 \) on \( 0 < x < l \). We do this by contradiction. If \( b_0 \psi(\kappa) + b_1 \psi'(\kappa) = 0 \) for some \( \kappa \) in \( 0 < \kappa < l \), then \( \psi(x) \) would satisfy

\[
\begin{align*}
\tag{16} (\phi(x)\psi')' + \lambda_1 f_u(x, y)\psi &= 0, \\
\tag{17} a_0 \psi(0) - a_1 \psi'(0) &= 0, \\
\tag{18} b_0 \psi(\kappa) + b_1 \psi'(\kappa) &= 0.
\end{align*}
\]

Now, from the usual variational characterization [2] of the principal eigenvalue of problems of the form of (13), (14), (15), we know that as the coefficient \( \rho(x) \) varies in one sense, the eigenvalue \( \lambda_1 \) varies in the opposite sense, and as the length of the interval varies in one sense, the eigenvalue \( \lambda_1 \) varies in the opposite sense. Thus, for fixed \( \lambda = \lambda_1 \), since \( f_u(x, y) < \rho(x) \), equation (18) can not hold for \( \kappa < l \). Hence, we conclude that \( b_0 \psi(x) + b_1 \psi'(x) > 0 \) on \( 0 < x < l \).
Finally, by once again using the fact that $\lambda_1(l)$ varies in the opposite sense from $l$, we conclude that if $\lambda < \lambda_1 \equiv \mu_1 \{ \rho \}$, then $l > 1$. Therefore, $\phi'(s) = b_0w(1) + b_1w'(1) > 0$. Q.E.D.

**Remark.** Actually, condition H-3 is not necessary for our proof. However, if $f(x, 0) \equiv 0$, the unique solution will be the trivial one. If $f(x, 0) = 0$, then the problem is closely related to one treated thoroughly by G. H. Pimbley [3]. Pimbley’s Theorem 1 gives uniqueness in the same range of $\lambda$. The extension to the case $f(x, 0) \neq 0$ is by no means trivial, however, and the consequences of this condition are pointed out in some detail in [4].

Recently, H. B. Keller and the present author [4] studied eigenvalue problems of a more general nature than (1), (2), (3) with regard to finding those values of $\lambda$ for which the problem has positive solutions, $u(x) > 0$. Such problems arise in the theory of nonlinear heat condition where the function $f(x, u)$ is always positive for $u \geq 0$. We defined the set $\{ \lambda \}$ of real values of $\lambda$ for which positive solutions exist as the spectrum of the problem, and the least upper bound of the spectrum was denoted by $\lambda^*$. For the problem (1), (2), (3) with $f$ positive and concave we proved that a unique positive solution exists for all $\lambda$ in $0 < \lambda < \lambda^*$, bounds were given on $\lambda^*$, and in the case that $\lim_{\phi \to x} [f_u(x, \phi)] = \rho(x) > 0$, we showed that $\lambda^* = \mu_1 \{ \rho \}$. Exact solutions of several simple problems involving convex nonlinearities show that more than one positive solution exists for all $\lambda$ in $0 < \lambda < \lambda^*$. Upon combining the results in [4] with the theorem of the present paper we can show that for positive convex nonlinearities $f(x, u)$ satisfying $\lim_{\phi \to x} [f_u(x, \phi)] = \rho(x) > 0$, a unique positive solution of (1), (2), (3) exists for all $\lambda$ in $0 < \lambda < \mu_1 \{ \rho \} \leq \lambda^*$. Thus, for $f$ convex a necessary condition for nonuniqueness is that $f_u(x, u)$ be unbounded in $u$.

**References**


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