A MEAN VALUE PROPERTY OF ELLIPTIC EQUATIONS WITH CONSTANT COEFFICIENTS

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The purpose of this note is to show that like harmonic functions which are characterized by Gauss mean value property (M.V.P.), general linear second order elliptic equation with real constant coefficients are also characterized by similar M.V.P. over ellipsoids.

In a domain R of E_n, consider the equation

$$\sum_{i,j=1}^{n} a_{ij}u_{x_ix_j} + 2\sum_{i=1}^{n} b_iu_{x_i} + cu = 0 \tag{1}$$

with real constant coefficients, where the real symmetric matrix $A = ((a_{ij}))$ is positive definite. Let $E(x_0, r)$ denote the interior of the ellipsoid $(x-x_0)A(x-x_0)^t = r^2$ (with center at the point x_0, $r>0$) in E_n.

THEOREM 1. *Let u_0 be a positive solution of* (1) *in $C^2(R)$. A function u in $C^2(R)$ is a solution of* (1) *if and only if u satisfies the M.V.P.*:

$$u(x_0)/u_0(x_0) = \int_{E(x_0,r)} u\exp(bAx^t)dx \Big/ \int_{E(x_0,r)} u_0\exp(bAx^t)dx \tag{2}$$

for each $E(x_0, r)$ whose closure lies in R, where $b = (b_1, \cdots, b_n)$.

PROOF. By a suitable affine transformation T, equation (1) can be transformed into the form

$$\Delta(u\exp(ky^t)) + (c - bAb^t)u\exp(ky^t) = 0 \tag{3}$$

where $k = bP^t$, P being the matrix of T and $P^tP = A$. Since $u_0(y)\exp(ky^t)$ satisfies (3) in $T(R)$, by Corollary 1 of [1], a function u in $C^2(R)$ is a solution of (1) in R if and only if the transformed function $u(y)/u_0(y)$ satisfies the M.V.P.

$$u(y_0)/u_0(y_0) = \int_{B(y_0,r)} u\exp(ky^t)dy \Big/ \int_{B(y_0,r)} u_0\exp(ky^t)dy \tag{4}$$

for each ball $B(y_0, r)$ whose closure lies in $T(R)$. That is if and only if u satisfies the M.V.P. (2) for each $E(x_0, r)$ whose closure lies in R.

REMARK. A positive solution of (1) always exists if $c - bAb^t \leqq 0$. Putting $c=0$ and $u_0=1$ we get, from Theorem 1, immediately

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THEOREM 2. *If $u \in C^2(R)$, then u is a solution of* (1) (*with* $c=0$) *if and only if u satisfies the M.V.P.*

$$u(x_0) = \int_{E(x_0,r)} u \exp(bAx^t)dx \Big/ \int_{E(x_0,r)} \exp(bAx^t)dx \tag{5}$$

for each $E(x_0, r)$ whose closure lies in R.

Setting $b=(0, \cdots, 0)$ and $A=I$, we get from (2) the M.V.P.

$$u(x_0)/u_0(x_0) = \int_{B(x_0,r)} u dx \Big/ \int_{B(x_0,r)} u_0 dx \tag{6}$$

which characterizes the equation (*) $\Delta u + cu = 0$. From (6) and (*) we get immediately the Gauss M.V.P. for harmonic functions by setting $c=0$ and $u_0=1$.

REFERENCE

1. Anil K. Bose, *Functions satisfying a weighted average property*, Trans. Amer. Math. Soc. 118 (1965), 472–487.

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