BASIC SETS OF POLYNOMIALS FOR GENERALIZED BELTRAMI AND EULER-POISSON-DARBOUX EQUATIONS AND THEIR ITERATES[1]

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1. **Introduction.** This paper concerns basic sets of polynomial solutions for the class of partial differential equations in m variables, $m \geqq 2$,

$$(1)\quad L_j^k(u) \equiv \left(D_m + (-1)^j \sum_{i=1}^{m-1} D_i \right)^k u = 0, \qquad j = 0, 1;\ k = 1, 2, \cdots,$$

where

$$D_i = \partial^2/\partial x_i^2 + (\alpha_i/x_i)(\partial/\partial x_i) \quad \text{with} \quad \alpha_i \geqq 0;\ i = 1, \cdots, m.$$

The iterated operators L_j^k are defined by the relations

$$L_j^{s+1}(u) = L_j[L_j^s(u)], \qquad s = 1, \cdots, k-1.$$

When $\alpha_1 = \cdots = \alpha_{m-1} = 0$ and $\alpha_m > 0$, $L_j(u) = 0$ is known as the Beltrami or the Euler-Poisson-Darboux (EPD) equation according as $j = 0$ or $j = 1$. If $\alpha_m = 0$ too, then $L_0(u) = 0$ and $L_1(u) = 0$ become the Laplace and wave equations, respectively. Basic sets of polynomial solutions for the Laplace and wave equations have been given in a number of papers [**1**]–[**5**]. In [**6**] Miles and Williams obtained basic sets of polynomials for the Beltrami and EPD equations from their result in [**3**]. In [**7**] the result of [**3**] was extended to form basic sets for the iterated Laplace and wave equations. Here we derive basic sets for (1) from the basic sets given in [**7**].

The Miles and Williams basic set of homogeneous polynomials of degree n for the k-fold iterated Laplace equation $\Delta^k u = 0$ $(\Delta = \sum_{i=1}^m \partial^2/\partial x_i^2)$ may be represented by

$$(2)\ H^n_{a_1, \cdots, a_m} = \sum_{j=0}^{[(n-a_m)/2]} (-1)^j \binom{j + [a_m/2]}{[a_m/2]} \Delta^j (x_1^{a_1} \cdots x_{m-1}^{a_{m-1}}) \frac{x_m^{2j+a_m}}{(2j+a_m)!},$$

where $a_1, \cdots, a_m$ are nonnegative integers such that $\sum_{i=1}^m a_i = n$ and $a_m \leqq 2k-1$. In particular, when $a_m = 0$, $H^n_{a_1, \cdots, a_{m-1}, 0}$ is harmonic,

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that is, it satisfies $\Delta u=0$. We shall prove that if for every index i $(1\leqq i\leqq n)$ such that $\alpha_i>0$ we restrict the a_i to be nonnegative even integers and replace $x_i^{2s_i}$ by

$$x_i^{(2s_i)} = \frac{1\cdot 3\cdots(2s_i-1)}{(1+\alpha_i)\cdots(2s_i-1+\alpha_i)}\, x_i^{2s_i}, \tag{3}$$

then (2) gives a basic set for $L_0^k(u)=0$ (or $L_1^k(u)=0$ if the factor $(-1)^j$ is deleted).

2. **Basic set for** $L_0^s(u)=0$, $s=1, 2, \cdots, k$. We first observe that any polynomial solution of (1) must be even in the variable x_i whenever $\alpha_i>0$, $1\leqq i\leqq m$. Indeed, suppose $\alpha_j>0$ and suppose that $u(x)$ is a polynomial solution of (1) which contains odd powers of x_j, $x=(x_1, \cdots, x_m)$. Let $P(x')$, a polynomial of $x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_m$, be the first nonvanishing coefficient of x_j^{2n+1} when $u(x)$ is arranged in ascending powers of x_j. Then the coefficient of $x_j^{2n+1-2k}$ in $L_0^k(u)=0$ would be $(2n+1)\cdots(2n-2k+3)(2n+\alpha_j)\cdots(2n+\alpha_j-2k+2)P(x')$, a nonvanishing function.

We assume that at least one of the α_i's is not zero. In fact, by changing subscripts if necessary, we can assume $\alpha_1=\cdots=\alpha_p=0$, $\alpha_{p+1}>0, \cdots, \alpha_m>0$, $0\leqq p\leqq m-1$. Let $a_1, \cdots, a_p, r_{p+1}, \cdots, r_m$ be a set of nonnegative integers satisfying the condition

$$\sum_{i=1}^{p} a_i + \sum_{i=p+1}^{m} 2r_i = N, \qquad r_m \leqq k-1, \tag{4}$$

and let

$$\begin{aligned} P^N_{a_1,\cdots,a_p,r_{p+1},\cdots,r_m} = \sum_{j=0}^{[(N-2r_m)/2]} (-1)^j \binom{j+r_m}{r_m} \\ \cdot \Delta^j(x_1^{a_1}\cdots x_p^{a_p} x_{p+1}^{2r_{p+1}} \cdots x_{m-1}^{2r_{m-1}}) \frac{x_m^{2j+2r_m}}{(2j+2r_m)!}. \end{aligned} \tag{5}$$

Denote by T_i $(p+1\leqq i\leqq m)$ the operator which replaces $x_i^{2s_i}$ by $x_i^{(2s_i)}$ (see (3)) in every term of the polynomial (5) and put $T=T_{p+1}\cdots T_m$.[2] Then the operator T applied to (5) replaces each factor $x_{p+1}^{2s_{p+1}}\cdots x_m^{2s_m}$ by $x_{p+1}^{(2s_{p+1})}\cdots x_m^{(2s_m)}$. Moreover, from the fact that $D_i\cdot T_i=T_i\partial^2/\partial x_i^2$ and $D_j\cdot T_k=T_k\cdot D_j$ for $j\neq k$, we see that

[2] We are indebted to the referee for suggesting this notation and pointing out that the operator T_i is a special case of a well-known operator treated by Lions, *Operateurs de Delsarte et problèmes mixtes*, Bull. Soc. Math. France **84** (1956); Proposition 2.1, p. 65.

$$
\begin{aligned}
L_0 \cdot T &= \left(\sum_{i=1}^{p} \partial^2/\partial x_i^2 + \sum_{i=p+1}^{m} D_i \right) T_{p+1} \cdots T_m \\
&= T \sum_{i=1}^{p} \partial^2/\partial x_i^2 + \sum_{i=p+1}^{m} T_{p+1} \cdots D_i T_i \cdots T_m \\
&= T \sum_{i=1}^{p} \partial^2/\partial x_i^2 + \sum_{i=p+1}^{m} T_{p+1} \cdots T_i \partial^2/\partial x_i^2 \cdots T_m \qquad (6) \\
&= T \sum_{i=1}^{p} \partial^2/\partial x_i^2 + T \sum_{i=p+1}^{m} \partial^2/\partial x_i^2 \\
&= T\Delta.
\end{aligned}
$$

Now let

$$
Q^N_{a_1,\cdots,a_p,r_{p+1},\cdots,r_m} = T(P^N_{a_1,\cdots,a_p,r_{p+1},\cdots,r_m}). \qquad (7)
$$

We assert that (7) forms a basic set for $L_0^k(u) = 0$.

LEMMA 1. $L_0(Q^N_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},0}) = 0$.

This follows from (6) and the fact that $P^N_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},0}$ is harmonic.

LEMMA 2. $L_0^s(Q^N_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},s}) = Q^{N-2s}_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},0}$, $0 \leqq 2s \leqq 2(k-1) \leqq N$.

Suppose that

$$
L_0^j(Q^N_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},j}) = Q^{N-2j}_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},0},
$$

$0 \leqq 2j < 2(k-1) \leqq N$, $a_1 + \cdots + a_p + 2r_{p+1} + \cdots + 2r_{m-1} + 2j = N$. Then for $a_1 + \cdots + a_p + 2r_{p+1} + \cdots + 2r_{m-1} + 2(j+1) = N$, we have

$$
\begin{aligned}
L_0^{j+1}(Q^N_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},j+1}) &= L_0^j[L_0(T(P^N_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},j+1}))] \\
&= L_0^j[T\Delta P^N_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},j+1}] \\
&= L_0^j[T(P^{N-2}_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},j})], \\
&\qquad\qquad \text{see (6) of [7]}, \\
&= L_0^j(Q^{N-2}_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},j}) \\
&= Q^{N-2-2j}_{a_1,\cdots,a_p,r_{p+1},\cdots,r_{m-1},0},
\end{aligned}
$$

where $a_1 + \cdots + a_p + 2r_{p+1} + \cdots + 2r_{m-1} + 2j = N-2$.

Now we verify that (7) forms a basic set for $L_0^k(u) = 0$. Consider first the case $2(k-1) \leqq N$. From Lemmas 1 and 2 it follows that all members of (7) satisfy the equation $L_0^k(u) = 0$. So we need only to show that (7) has the correct number of independent polynomials. For a given integer $N \geqq 0$, it is clear that (7) has as many independent polynomials as there are distinct ways of choosing the set $a_1, \cdots, a_p, r_{p+1}, \cdots, r_m$ which satisfies (4). Let

$$u = \sum A_{a_1, \cdots, a_p, r_{p+1}, \cdots, r_m} x_1^{a_1} \cdots x_p^{a_p} x_{p+1}^{(2r_{p+1})} \cdots x_m^{(2r_m)},$$

$a_1 + \cdots + a_p + 2r_{p+1} + \cdots + 2r_m = N$, $0 \leqq p \leqq m-1$, be any homogeneous polynomial of $x_1, \cdots, x_m$ of degree N, $N \geqq 2(k-1)$, which is even in the variables x_i, $p+1 \leqq i \leqq m$. Here we have already replaced each $x_i^{2s_i}$ by $x_i^{(2s_i)}$. Then every coefficient $A_{a_1, \cdots, a_p, r_{p+1}, \cdots, r_m}$ of u can be represented, apart from constant factor, as

$$A_{a_1, \cdots, a_p, r_{p+1}, \cdots, r_m} \sim (d_1^{a_1} \cdots d_p^{a_p} D_{p+1}^{r_{p+1}} \cdots D_m^{r_m}) u,$$

where $d_i = \partial / \partial x_i$, $i = 1, 2, \cdots, p$. If $L_0^k(u) = 0$, so that

$$D_m^k u = -\left(\sum \frac{k!}{a_1! \cdots a_p! r_{p+1}! \cdots r_m!} d_1^{2a_1} \cdots d_p^{2a_p} D_{p+1}^{r_{p+1}} \cdots D_m^{r_m} \right) u$$

where $a_1 + \cdots + a_p + r_{p+1} + \cdots + r_m = k$ with $r_m \leqq k-1$, then every derivative of the form $(d_1^{a_1} \cdots d_p^{a_p} D_{p+1}^{r_{p+1}} \cdots D_m^{r_m}) u$ can be written in such a way that D_m occurs no more than $(k-1)$ times. Thus, if $L_0^k(u) = 0$, all coefficients of u are linear combinations of the coefficients $A_{s_1, \cdots, s_p, t_{p+1}, \cdots, t_{m-1}, z}$, $0 \leqq z \leqq k-1$, where $s_1 + \cdots + s_p + 2t_{p+1} + \cdots + 2z = N$ which coincides with (4). Therefore, for $2(k-1) \leqq N$, the set (7) is correctly numbered and hence forms a basic set for $L_0^k(u) = 0$.

In the case $N < 2(k-1)$, it is clear that all the members of (7) satisfy $L_0^k(u) = 0$. In order to prove that (7) has the correct number of polynomials, we examine (4) with $r_m \leqq [N/2]$ under the following cases.

Case 1. $N = 2n$, $n \geqq 0$.

Suppose first that $p = 2q$; then for each s, $0 \leqq s \leqq n$, where s replaces r_m, only $2v$ of the a_i's can be chosen as odd integers with $0 \leqq v \leqq [q, n]$. Here $[q, n]$ denotes the smaller of the integers q and n. Hence, writing $a_i = 2r_i$ if a_i is even and $a_i = 2r_i + 1$ if a_i is odd $(1 \leqq i \leqq p)$, we have $a_1 + \cdots + a_p = 2r_1 + \cdots + 2r_p + 2v$ so that (4) becomes $r_1 + \cdots + r_{m-1} = n - s - v$. Now for each s, $0 \leqq s \leqq n$, the set (7) has

$$\sum_{v=0}^{[q,n]} \binom{2q}{2v} \binom{m+n-2-s-v}{m-2}$$

independent polynomials, since for each of the

$$\binom{2q}{2v}$$

ways of choosing $2v$ of the a_i's odd the set $P^{2n}_{r_1,\cdots,r_{m-1},s}$ is seen to be generated by the monomials $x_1^{a_1}\cdots x_p^{a_p}x_{p+1}^{2r_{p+1}}\cdots x_{m-1}^{2r_{m-1}}$ with $a_1+\cdots+a_p=2r_1+\cdots+2r_p+2v$, where $x_1^{2r_1}\cdots x_{m-1}^{2r_{m-1}}$ are the individual terms appearing in $(\sum_{i=1}^{m-1}x_i^2)^{n-s-v}$. Therefore, for $N=2n$ and $p=2q$, the set (7) consists of

$$\sum_{s=0}^{n}\sum_{v=0}^{[q,n]}\binom{2q}{2v}\binom{m+n-2-s-v}{m-2} \tag{8}$$

independent polynomials homogeneous of degree $2n$. Here

$$\binom{a}{b}$$

is interpreted as zero whenever $a<b$.

If $p=2q+1$, then (7) will have

$$\sum_{s=0}^{n}\sum_{v=0}^{[q,n]}\binom{2q+1}{2v}\binom{m+n-2-s-v}{m-2} \tag{9}$$

independent polynomials homogeneous of degree $2n$.

Case 2. $N=2n+1$, $n\geqq 0$.

In this case $p\neq 0$. This means that no polynomial of odd degree can satisfy $L_0^k(u)=0$ when $\alpha_i>0$, $i=1, 2, \cdots, m$. Again, suppose first that $p=2q$; then $2v+1$ of the a_i's must be chosen odd, $0\leqq v\leqq[q-1, n]$. Hence we can write $a_1+\cdots+a_p=2r_1+\cdots+2r_p+2v+1$ and again (4) reduces to $r_1+\cdots+r_{m-1}=n-s-v$. By the same argument as in Case 1, we see that (7) has

$$\sum_{s=0}^{n}\sum_{v=0}^{[q-1,n]}\binom{2q}{2v+1}\binom{m+n-2-s-v}{m-2} \tag{10}$$

independent homogeneous polynomials of degree $2n+1$.

For $p=2q+1$, (7) consists of

$$\sum_{s=0}^{n}\sum_{v=0}^{[q,n]}\binom{2q+1}{2v+1}\binom{m+n-2-s-v}{m-2} \tag{11}$$

independent homogeneous polynomials.

Noting that for fixed v the summands corresponding to $s>n-v$ are all zero, we can carry out the summation with respect to s in the formulas (8)–(11). In fact, from (8) for example, we obtain

$$\sum_{s=0}^{n}\sum_{v=0}^{[q,n]}\binom{2q}{2v}\binom{m+n-2-s-v}{m-2}=\sum_{v=0}^{[q,n]}\binom{2q}{2v}$$
$$\cdot\sum_{s=0}^{n-v}\binom{m+n-2-s-v}{m-2}=\sum_{v=0}^{[q,n]}\binom{2q}{2v}\binom{m+n-v-1}{m-1},$$

the number of independent polynomials in (7) when $N=2n$, $n<k-1$, and $p=2q$. Indeed, when $N<2(k-1)$ the basic set elements in case $N=2n$ and $p=2q$ could be chosen for each v, $0\leqq v\leqq[q,n]$, as the individual monomials $x_1^{a_1}\cdots x_p^{a_p}x_{p+1}^{2r_{p+1}}\cdots x_m^{2r_m}$, $a_1+\cdots+a_p=2r_1+\cdots+2r_p+2v$, where $x_1^{2r_1}\cdots x_m^{2r_m}$ are the individual terms appearing in $(\sum_{i=1}^m x_i^2)^{n-v}$ which has precisely

$$\binom{m+n-v-1}{m-1}$$

elements. Members of basic set for the other cases can also be chosen in the same manner thus proving that (7) is correctly numbered.

Therefore, for given integers $N\geqq 0$, $k\geqq 1$, and the associated sets of nonnegative integers $a_1,\cdots,a_p$, $r_{p+1},\cdots,r_m$ satisfying (4), ($2r_m\leqq N$ if $N<2(k-1)$), the set of polynomials given by (7) is a basic set for $L_0^k(u)=0$.

3. **Basic set for** $L_1^k(u)=0$. The corresponding basic set of polynomials for $L_1^k(u)=0$ under the same assumption on the α_i's as before may be deduced from (7) upon replacement of x_m by ix_m. We have the same formula as (7) except for the absence of the factor $(-1)^j$. This fact can of course be established by the same procedure as in §2 using (5) with the factor $(-1)^j$ deleted.

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