ON STRONG RIESZ SUMMABILITY FACTORS OF INFINITE SERIES. I

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(1.1) Let $\sum_{n=1}^{\infty} a_n$ be a given infinite series, and $\{\lambda_n\}$ an increasing sequence of positive numbers, tending to infinity with n. We write

$$A^0(\omega) = A(\omega) = \sum_{\lambda_n<\omega} a_n,$$

$$A^k(\omega) = \sum_{\lambda_n<\omega} (\omega - \lambda_n)^k a_n$$

$$= \int_0^{\omega} (\omega - t)^k dA(t),$$

$$A^k(\omega) = 0 \quad \text{for } \omega \leqq 1, \text{ and } k > -1.$$

The series $\sum a_n$ is said to be summable (R, λ, k) to the sum s, if $\lim \omega^{-k}A^k(\omega)\rightarrow s$ as $\omega\rightarrow\infty$. The given series is said to be strongly summable (R, λ, k), or simply summable $[R, \lambda, k]$ to the sum s, if

$$\int_0^{\omega} | x^{-k+1}A^{k-1}(x) - s | dx = o(\omega);$$

where $k>0$. The series $\sum a_n$ is said to be strongly summable (R, λ, k) with index $m>0$, or summable $[R, \lambda, k, m]$ to the sum s, if

$$\int_0^{\omega} | x^{-k+1}A^{k-1}(x) - s |^m dx = o(\omega),$$

where $k>0$ and $km'>1$, $(1/m+1/m'=1)$.

(1.2) The classical second theorem of consistency due to Hardy and Riesz [3] is to the effect that if $\sum a_n$ is summable (R, λ, k) and $\lambda_n=e^{\mu_n}$, then it is also summable (R, μ, k) to the same sum. Later Hardy [2] generalized this theorem and proved:

THEOREM A. *If the series $\sum a_n$ is summable (R, λ, k) and μ is a logarithmico-exponential function (briefly an L-function) of λ, tending to infinity with λ, such that $\mu=O(\lambda^{\Delta})$, where Δ is some constant, then $\sum a_n$ is summable (R, μ, k).*

Finally Hirst [4] removed the limitation on μ of being an L-function, replacing μ by a more general function $\phi(t)$, and proved:

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THEOREM B. *If* $\sum a_n$ *is summable* (R, λ, k), *then it is summable* (R, μ, k) *to the same sum, where* $\mu=\phi(\lambda)$, *and* $\phi(t)$ *is a function which increases steadily to infinity with t and is a* $(k+1)$*th indefinite integral for* $t>0$, *such that*

$$\int_0^x t^n \left| \phi^{(n+1)}(t) \right| dt = O\{\phi(x)\}; \qquad n = 1, 2, \cdots, k.$$

The following theorem on strong Riesz summability factors was proved by Borwein and Shawyer [**1**].

THEOREM C. *For all* $k \geqq 1$, *if*

(i) $\phi(t)$ *is an L-function*,

(ii) $1/\omega = O\{\phi'(\omega)/\phi(\omega)\}$,

(iii) $\psi(\omega) = \{\phi(\omega)/\omega\phi'(\omega)\}^k$,

then $\sum a_n\psi(\lambda_n)$ *is summable* $[R, \phi(\lambda), k]$ *whenever* $\sum a_n$ *is summable* $[R, \lambda, k]$.

The object of this paper is to establish a more general summability factor theorem for strong Riesz summability, which includes as a particular case Theorem C.

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(2.1) In the following we take functions $\phi(t)$ and $\psi(t)$ to be defined in $(0, \infty)$ and to be as many times differentiable as required. In addition let $\phi(t)$ be nonnegative, monotone increasing, and tending to infinity with t.

We establish the following theorem.

THEOREM. *Let k be a positive integer, and* $\phi(t)$ *and* $\psi(t)$ *be* $(k+1)$*th indefinite integrals for* $t>0$. *If there is a positive, nondecreasing function* $\gamma(t)$ *such that*

(i) $\gamma(t) = O(t)$ *in* (a, ∞); $a>0$,

(ii) $t^k\psi^{(n)}(t) = O(\{\gamma(t)\}^{k-n})$; $n=0, 1, \cdots, k$; $t>a$,

(iii) $\{\gamma(t)\}^n\phi^{(n)}(t) = O[\phi(t)]$: $n=1, 2, \cdots, k$; $t>a$,

then $\sum a_n\psi(\lambda_n)$ *is summable* $[R, \phi(\lambda), k, m]$, *whenever* $\sum a_n$ *is summable* $[R, \lambda, k, m]$, *where* $m \geqq 1$ *and* $km' > 1$.

(2.2) The following lemmas will be required in the proof of our theorem.

LEMMA 1 [**3**]. *If k is a positive integer, then* $A(t) = (1/k!)(d/dt)^k A^k(t)$.

LEMMA 2 [**5**]. *If* $\sum a_n$ *is summable* $[R, \lambda, k, m]$ *for* $m \geqq 1$, *then it is summable* (R, λ, k).

LEMMA 3 [8]. *If n is a positive integer and $m\neq 0$, then the nth derivative of $\{f(x)\}^m$ is a sum of the constant multiples of a finite number of terms of the form*

$$\{f(x)\}^{m-q}\prod_{p=1}^{n}\{f^{(p)}(x)\}^{\alpha_p},$$

where $1\leqq q\leqq n$ and the α's are nonnegative integers such that

$$\sum_1^n \alpha_p = q \quad and \quad \sum_1^n p\alpha_p = n.$$

If m is a positive integer, then $1\leqq q\leqq \min(m, n)$.

3. **Proof of the theorem.** We may suppose without loss of generality that the sum of the given series is zero. Then the hypothesis of summability $[R, \lambda, k, m]$ of $\sum a_n$ reduces to

$$\int_0^\omega | A^{k-1}(t)|^m dt = o[\omega^{(k-1)m+1}]. \tag{3.1}$$

And for summability $[R, \phi(\lambda), k, m]$ of $\sum a_n\psi(\lambda_n)$ we must show

$$\int_0^\omega \phi'(t) | B^{k-1}\{\phi(t)\}|^m dt = o[\{\phi(\omega)\}^{(k-1)m+1}], \tag{3.2}$$

where

$$B^{k-1}\{\phi(t)\} = \int_0^t \{\phi(t) - \psi(u)\}^{k-1}\psi(u)dA(u). \tag{3.3}$$

Integrating by parts, we have

$$B^{k-1}\{\phi(t)\} = -\int_0^t (\partial/\partial u)[\{\phi(t) - \phi(u)\}^{k-1}\psi(u)]A(u)du.$$

And by Lemma 1,

$$\begin{aligned} &B^{k-1}\{\phi(t)\} \\ &= -\frac{1}{(k-1)!}\int_0^t \frac{\partial}{\partial u}[\{\phi(t) - \phi(u)\}^{k-1}\psi(u)]\left(\frac{d}{du}\right)^{k-1}A^{k-1}(u)du. \end{aligned} \tag{3.4}$$

Since $A^k(u)$ and its first $(k-1)$ derivatives vanish at $u=0$, integrating (3.4) by parts $(k-1)$ times, we get

$$\begin{aligned} B^{k-1}\{\phi(t)\} &= A^{k-1}(t)\psi(t)\{\phi'(t)\}^{k-1} \\ &+ \frac{(-1)^k}{(k-1)!}\int_0^t A^{k-1}(u)\left(\frac{\partial}{\partial u}\right)^k[\{\phi(t) - \phi(u)\}^{k-1}\psi(u)]du. \end{aligned} \tag{3.5}$$

Thus to prove the theorem, it is sufficient to prove that

$$(3.6)\quad \int_0^\omega |A^{k-1}(t)|^m\{\psi(t)\}^m\{\phi'(t)\}^{(k-1)m+1}dt = o[\{\phi(\omega)\}^{(k-1)m+1}]$$

and

$$(3.7)\quad \int_0^\omega \phi'(t)\left|\int_0^t A^{k-1}(u)\left(\frac{\partial}{\partial u}\right)^k[\{\phi(t)-\phi(u)\}^{k-1}\psi(u)]du\right|^m dt$$
$$= o[\{\phi(\omega)\}^{(k-1)m+1}].$$

Proof of (3.6). Let

$$J = \int_0^\omega |A^{k-1}(t)|^m\{\psi(t)\}^m\{\phi'(t)\}^{(k-1)m+1}dt$$
$$= \int_0^1 + \int_1^\omega .$$

By hypotheses (ii) and (iii) of the theorem and the fact that $A^k(\omega)=0$ for $\omega \leqq 1$ and $k > -1$, we have

$$J = O\left(\int_1^\omega |A^{k-1}(t)|^m\left(\frac{\gamma(t)}{t}\right)^{km}\left(\frac{\phi(t)}{\gamma(t)}\right)^{(k-1)m+1}dt\right)$$
$$= O\left(\{\phi(\omega)\}^{(k-1)m+1}\int_1^\omega |A^{k-1}(t)|^m t^{-km}\{\gamma(t)\}^{m-1}dt\right)$$
$$= O\left(\{\phi(\omega)\}^{(k-1)m+1}\{\gamma(\omega)\}^{m-1}\int_1^\omega |A^{k-1}(t)|^m t^{-km}dt\right),$$

since $k \geqq 1$, $m \geqq 1$ and $\gamma(t)$ is a nondecreasing positive function.

Integrating the last expression by parts and using (3.1) we have:

$$J = o[\{\phi(\omega)\}^{(k-1)m+1}\{\gamma(\omega)\}^{m-1}\omega^{1-m}]$$
$$= o[\{\phi(\omega)\}^{(k-1)m+1}]$$

by (i) of the theorem.

Proof of (3.7). Note that $(\partial/\partial u)^k[\{\phi(t)-\phi(u)\}^{k-1}\psi(u)]$ is a sum of constant multiples of terms of type

$$\psi^{(\nu)}(u)(\partial/\partial u)^{k-\nu}\{\phi(t)-\phi(u)\}^{k-1};\qquad 0 \leqq \nu \leqq k,$$

and, by Lemma 3, $(\partial/\partial u)^{k-\nu}\{\phi(t)-\phi(u)\}^{k-1}$ is a linear combination of expressions of the form

$$\{\phi(t)-\phi(u)\}^{k-1-\mu}\prod_{p=1}^{k-\nu}\{\phi^{(p)}(u)\}^{\alpha_p},$$

where $0 \leqq \mu \leqq k-\nu-1$ and

$$\sum_{p=1}^{k-\nu} \alpha_p = \mu, \qquad \sum_{p=1}^{k-\nu} p\alpha_p = k-\nu.$$

Thus to prove (3.7) it is enough to show that

$$(3.8)\quad \int_0^\omega \phi'(t) \left| \int_0^t A^{k-1}(u)\{\phi(t)-\phi(u)\}^{k-1-\mu}\psi^{(\nu)}(u) \prod_{p=1}^{k-\nu} \{\phi^{(p)}(u)\}^{\alpha_p} du \right|^m dt$$
$$= o[\{\phi(\omega)\}^{(k-1)m+1}].$$

Consider

$$(3.9)\quad f(t) = \int_0^t A^{k-1}(u)\{\phi(t) - \phi(u)\}^{k-1-\mu}\psi^{(\nu)}(u) \prod_{p=1}^{k-\nu} \{\phi^{(p)}(u)\}^{\alpha_p} du.$$

Integrating (3.9) by parts and making use of the fact that summability $[R, \lambda, k, m]$ implies summability (R, λ, k) we have

$$f(t) = \left[o(u^k)\psi^{(\nu)}(u)\{\phi(t) - \phi(u)\}^{k-1-\mu} \prod_{p=1}^{k-\nu} \{\phi^{(p)}(u)\}^{\alpha_p} \right]_0^t$$
$$+ o\left(\int_0^t u^k \frac{d}{du} [\psi^{(\nu)}(u)\{\phi(t) - \phi(u)\}^{k-1-\mu} \prod_{p=1}^{k-\nu} \{\phi^{(p)}(u)\}^{\alpha_p}] du \right)$$
$$= I_1 + I_2, \text{ say.}$$

When $0 \leqq \mu < k-1$, then $I_1 = 0$; and when $\mu = k-1$,

$$I_1 = \left[o(t^k)\psi^{(\nu)}(t) \prod_{p=1}^{k-\nu} \{\phi^{(p)}(t)\}^{\alpha_p} \right]$$

Using hypotheses (ii) and (iii) and the relations

$$\sum_1^{k-\nu} p\alpha_p = k-\nu, \qquad \sum_1^{k-\nu} \alpha_p = \mu$$

we get

$$I_1 = o\left(\prod_{p=1}^{k-\nu} [\{\gamma(t)\}^p\{\phi^{(p)}(t)\}]^{\alpha_p} \right)$$
$$= o(\{\phi(t)\}^{\Sigma\alpha_p})$$
$$= o(\{\phi(t)\}^{\mu}).$$

Thus

$$(3.10)\qquad I_1 = o(\{\phi(t)\}^{k-1}).$$

Now $I_2=I_3+I_4+I_5$, where

$$I_3 = o\left(\int_0^t u^k\{\phi(t) - \phi(u)\}^{k-2-\mu}\phi'(u)\psi^{(\nu)}(u)\prod_{p=1}^{k-\nu}\{\phi^{(p)}(u)\}^{\alpha_p}du\right),$$

$$I_4 = o\left(\int_0^t u^k\{\phi(t) - \phi(u)\}^{k-1-\mu}\psi^{(\nu+1)}(u)\prod_{p=1}^{k-\nu}\{\phi^{(p)}(u)\}^{\alpha_p}du\right),$$

$$I_5 = o\left(\int_0^t u^k\{\phi(t) - \phi(u)\}^{k-1-\mu}\psi^{(\nu)}(u)\sum_{p=1}^{k-\nu}\alpha_p\frac{\phi^{(p+1)}(u)}{\phi^p(u)}\prod_{r=1}^{k-\nu}\{\phi^{(r)}(u)\}^{\alpha_r}du\right).$$

By hypothesis (ii),

$$I_3 = o\left(\int_0^t\{\phi(t) - \phi(u)\}^{k-\mu-2}\phi'(u)\prod_{p=1}^{k-\nu}[\{\gamma(u)\}^p\phi^{(p)}(u)]^{\alpha_p}du\right)$$
$$= o\left(\{\phi(t)\}^{\mu}\int_0^t\{\phi(t) - \phi(u)\}^{k-\mu-2}\phi'(u)du\right).$$

Thus

$$(3.11)\qquad I_3 = o(\{\phi(t)\}^{k-1}).$$

Consider I_4 (a) If all α_p's are zero then $\mu=0$ and $k=\nu$, since $\sum\alpha_p=\mu$ and $\sum p\alpha_p=k-\nu$. Thus

$$I_4 = o\left(\int_0^t u^k\{\phi(t) - \phi(u)\}^{k-1}\psi^{(k+1)}(u)du\right)$$
$$= o\left(t^k\{\phi(t)\}^{k-1}\int_0^t\psi^{(k+1)}(u)du\right)$$
$$= o(\{\phi(t)\}^{k-1}),$$

since $t^k\psi^{(k)}(t)=O(1)$ by (ii).

(b) Suppose that not all α_p's are zero, so that $\mu\geqq 1$. If $\alpha_q\geqq 1$, we have, by (ii) and (iii)

$$I_4 = o\left(\{\phi(t)\}^{\mu-1}\int_0^t\{\phi(t) - \phi(u)\{^{k-1-\mu}\{\gamma(u)\}^{q-1}\phi^{(q)}(u)du\right)$$
$$= o\left(\{\phi(t)\}^{k-2}\{\gamma(t)\}^{q-1}\int_0^t\phi^{(q)}(u)du\right)$$
$$= o(\{\phi(t)\}^{k-2}[\{(t)\}^{q-1}\phi^{(q-1)}(t)])$$
$$= o(\{\phi(t)\}^{k-1}).$$

Thus

$$I_4 = o(\{\phi(t)\}^{k-1}). \tag{3.12}$$

When $k>1$,

$$I_5 = o\left(\int_0^t \{\phi(t)-\phi(u)\}^{k-1-\mu} \sum_{p=1}^{k-\nu} \alpha_p\{\gamma(u)\}^p \left|\phi^{(p+1)}(u)\right| \cdot \prod_{r=1}^{k-\nu}{}^{(p)} \left|\{\gamma(u)\}^r\phi^{(r)}(u)\right|^{\alpha_r} du\right),$$

where $\prod^{(p)}$ means that in the pth factor the power is α_p-1 (not α_p). Hence

$$\begin{aligned} I_5 &= o\left(\int_0^t \{\phi(t)-\phi(u)\}^{k-1-\mu} \sum_{p=1}^{k-\nu} \{\gamma(u)\}^p \phi^{(p+1)}(u)\{\phi(u)\}^{\Sigma\alpha_p-1}du\right) \\ &= o\left(\{\phi(t)\}^{k-2}\sum_{p=1}^{k-\nu}\{\gamma(t)\}^p\int_0^t \phi^{(p+1)}(u)du\right) \\ &= o\left(\{\phi(t)\}^{k-2}\sum_{p=1}^{k-\nu}\left[\{\gamma(t)\}^p\phi^{(p)}(t)\right]\right) \\ &= o(\{\phi(t)\}^{k-1}). \end{aligned}$$

When $k=1$, the proof is similar to that for I_4, case (a). Thus

$$I_5 = o(\{\phi(t)\}^{k-1}). \tag{3.13}$$

From (3.10), (3.11), (3.12) and (3.13) it follows that the function $f(t)$ of (3.9) is $o(\{\phi(t)\}^{k-1})$. Thus

$$\begin{aligned} \int_0^\omega \phi'(t)\left|f(t)\right|^m dt &= o\left(\int_0^\omega \{\phi(t)\}^{(k-1)m}\phi'(t)dt\right) \\ &= o\left[\{\phi(\omega)\}^{(k-1)m+1}\right]. \end{aligned}$$

This completes the proof of the theorem.

In the special case when $\psi(t)=1$, $\gamma(t)=t$, $m=1$, we have the the "second theorem of consistency for strong summability" due to Srivastava [**6**].

When $\phi(t)=e^t$, $\psi(t)=t^{-k}$, $\gamma(t)=1$, $m=1$, we have the following strong Riesz summability analogue of a theorem due to Tatchell [**7**].

THEOREM. *If k is a positive integer and the series $\sum a_n$ is summable $[R, \lambda, k]$, then $\sum a_n\lambda_n^{-k}$ is summable $[R, e^\lambda, k]$.*

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