ON A THEOREM OF HAAR

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Let \( \{ \phi_n \} \) be an orthonormal set in \( L^2[0, 1] \), \( \phi_n \) having finite total variation \( v_n \). Haar proved \([1]\) that \( \lim v_n = \infty \) and that consequently \( \{ \phi_n \} \) has an intrinsic ordering according to nondecreasing total variation. He showed, in fact, that in this ordering \( v_n > c \sqrt{n} \) where \( c \) is an absolute constant. A different proof was later given by Rudin \([2]\).

We present a short proof that \( \lim v_n = \infty \) under a much weaker hypothesis than orthonormality.

**Theorem.** Let \( \{ \phi_n \}_{1}^{\infty} \) be a set of functions in \( L^2[0, 1] \) such that for each \( n \), \( \| \phi_n \| = 1 \) and \( \phi_n \) has finite total variation \( v_n \). If

\[
\lim \sup_{n} \lim \sup_{k} (\phi_n, \phi_k) < 1,
\]

then

\[
\lim v_n = \infty.
\]

**Proof.** Assume (2) is false. Then there is a subsequence \( \{ \phi_{n_i} \} \) and a constant \( M \) such that \( v_{n_i} < M \) for each \( i \). Since

\[
\sup_x \phi_{n_i}(x) - \inf_x \phi_{n_i}(x) \leq v_{n_i} < M,
\]

it follows that \( |\phi_{n_i}(x)| \leq M + 1 \) for all \( x \). For, if \( \sup_x \phi_{n_i}(x) > M + 1 \), then

\[
\inf_x \phi_{n_i}(x) > \sup_x \phi_{n_i}(x) - M > 1
\]

which is incompatible with the hypothesis \( \| \phi_{n_i} \| = 1 \). Similarly, \( \inf_x \phi_{n_i}(x) > -(M + 1) \).

Thus the functions \( \phi_{n_i} \) are uniformly bounded and their total variations are bounded. By Helly’s Theorem, there is a convergent subsequence \( \{ \phi_{n_{i_k}} \} \).

\[
(\phi_{n_{i_j}}, \phi_{n_{i_k}}) = \frac{1}{2}(\| \phi_{n_{i_j}} \|^2 + \| \phi_{n_{i_k}} \|^2 - \| \phi_{n_{i_j}} - \phi_{n_{i_k}} \|^2)
\]

\[
= 1 - \frac{1}{2}\| \phi_{n_{i_j}} - \phi_{n_{i_k}} \|^2.
\]

By bounded convergence, \( \| \phi_{n_{i_j}} - \phi_{n_{i_k}} \|^2 \to 0 \) as \( j, k \to \infty \). Therefore, given \( \epsilon \),

\[
(\phi_{n_{i_j}}, \phi_{n_{i_k}}) > 1 - \epsilon
\]

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for all sufficiently large values of \( j \) and \( k > j \). Hence,

\[
\limsup_{n} (\phi_{n,j}, \phi_{k}) > 1 - \epsilon
\]

for infinitely many indices \( n_{i,j} \). Since \( \epsilon \) was arbitrary,

\[
\limsup_{n} \limsup_{k} (\phi_{n}, \phi_{k}) = 1.
\]

But this contradicts hypothesis (1) and the theorem is proved.

Haar’s result that \( v_{n} > c \sqrt{n} \) for an orthonormal set can also be extended.

**Theorem.** If

\[
\sum_{1 \leq i < j < \infty} |(\phi_{i}, \phi_{j})| < \infty,
\]

then \( \{\phi_{n}\} \) can be ordered according to nondecreasing total variations so that \( v_{n} > a \sqrt{n} \) where \( a \) is a constant depending only on the sum (3).

For the proof, one need only modify Haar’s original argument by using in place of the Bessel inequality the inequality

\[
c_{1}^{2} + c_{2}^{2} + \cdots + c_{n}^{2} \leq b||f||^{2}, \quad c_{i} = (f, \phi_{i})
\]

which holds under hypothesis (3).

Condition (1) was suggested by the referee to replace our somewhat stronger hypothesis, \( \limsup_{n,k} (\phi_{n}, \phi_{k}) = 1 \).

**References**