ON A THEOREM OF HAAR

J. J. PRICE

Let $\{\phi_n\}$ be an orthonormal set in $L^2[0, 1]$, ϕ_n having finite total variation v_n. Haar proved [1] that $\lim v_n = \infty$ and that consequently $\{\phi_n\}$ has an intrinsic ordering according to nondecreasing total variation. He showed, in fact, that in this ordering $v_n > c\sqrt{n}$ where c is an absolute constant. A different proof was later given by Rudin [2].

We present a short proof that $\lim v_n = \infty$ under a much weaker hypothesis than orthonormality.

THEOREM. *Let $\{\phi_n\}_1^\infty$ be a set of functions in $L^2[0, 1]$ such that for each n, $\|\phi_n\| = 1$ and ϕ_n has finite total variation v_n. If*

$$\limsup_n \limsup_k (\phi_n, \phi_k) < 1, \tag{1}$$

then

$$\lim v_n = \infty. \tag{2}$$

PROOF. Assume (2) is false. Then there is a subsequence $\{\phi_{n_i}\}$ and a constant M such that $v_{n_i} < M$ for each i. Since

$$\sup_x \phi_{n_i}(x) - \inf_x \phi_{n_i}(x) \leqq v_{n_i} < M,$$

it follows that $|\phi_{n_i}(x)| \leqq M+1$ for all x. For, if $\sup_x \phi_{n_i}(x) > M+1$, then

$$\inf_x \phi_{n_i}(x) > \sup_x \phi_{n_i}(x) - M > 1$$

which is incompatible with the hypothesis $\|\phi_{n_i}\| = 1$. Similarly, $\inf_x \phi_{n_i}(x) > -(M+1)$.

Thus the functions ϕ_{n_i} are uniformly bounded and their total variations are bounded. By Helly's Theorem, there is a convergent subsequence $\{\phi_{n_{i_k}}\}$.

$$\begin{aligned}(\phi_{n_{i_j}}, \phi_{n_{i_k}}) &= \tfrac{1}{2}(\|\phi_{n_{i_j}}\|^2 + \|\phi_{n_{i_k}}\|^2 - \|\phi_{n_{i_j}} - \phi_{n_{i_k}}\|^2) \\ &= 1 - \tfrac{1}{2}\|\phi_{n_{i_j}} - \phi_{n_{i_k}}\|^2.\end{aligned}$$

By bounded convergence, $\|\phi_{n_{i_j}} - \phi_{n_{i_k}}\|^2 \to 0$ as $j, k \to \infty$. Therefore, given ϵ,

$$(\phi_{n_{i_j}}, \phi_{n_{i_k}}) > 1 - \epsilon$$

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for all sufficiently large values of j and $k>j$. Hence,

$$\limsup(\phi_{n_{i_j}}, \phi_k) > 1 - \epsilon$$

for infinitely many indices n_{i_j}. Since ϵ was arbitrary,

$$\limsup_n \limsup_k(\phi_n, \phi_k) = 1.$$

But this contradicts hypothesis (1) and the theorem is proved.

Haar's result that $v_n > c\sqrt{n}$ for an orthonormal set can also be extended.

THEOREM. *If*

$$\sum_{1 \leq i < j < \infty}^{\infty} |(\phi_i, \phi_j)| < \infty, \tag{3}$$

then $\{\phi_n\}$ *can be ordered according to nondecreasing total variations so that* $v_n > a\sqrt{n}$ *where* a *is a constant depending only on the sum* (3).

For the proof, one need only modify Haar's original argument by using in place of the Bessel inequality the inequality

$$c_1^2 + c_2^2 + \cdots + c_n^2 \leqq b\|f\|^2, \qquad c_i = (f, \phi_i)$$

which holds under hypothesis (3).

Condition (1) was suggested by the referee to replace our somewhat stronger hypothesis, $\limsup_{n,k}(\phi_n, \phi_k) = 1$.

REFERENCES

1. A. Haar, *Über einige Eigenschaften der orthogonalen Funktionensysteme*, Math. Z. **31** (1930), 128–137.

2. Walter Rudin, *L^2-approximation by partial sums of orthogonal developments*, Duke Math. J. **19** (1952), 1–4.

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