EXISTENCE OF FINITE INVARIANT MEASURES FOR MARKOV PROCESSES

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Let x_t be a Markov process on a locally compact metric space $(\mathcal{E}, \rho)$ with a countable base. Let $\mathcal{B}$ be the σ-algebra generated by the open sets, and for $A \in \mathcal{B}$, $t > 0$, set $P(t, x, A) = \Pr\{x_t \in A \mid x_0 = x\}$. Khasminskii [1] has related, in a natural way, the existence of a finite invariant measure for x_t with the mean time to hit a compact set. Our object is to relate the existence of such a measure to properties of the measures $P(t, x, \cdot)$, $t > 0$, $x \in \mathcal{E}$.

We assume that for fixed t and A, $P(t, x, A)$ is continuous in x, that for an ϵ-neighborhood $S_\epsilon(x)$ about x, $P(t, x, S_\epsilon(x)) \to 1$ as $t \to 0$ uniformly on compacta, and that $P(t, x, \mathcal{E}) = 1$. Let C^+ be the strictly positive cone of the Banach space $\mathrm{ca}(\mathcal{E}, \mathcal{B})$ consisting of the countably additive set functions on $\mathcal{B}$ with variation norm. Let U_t, $t \geqq 0$, be the semigroup defined for $\mu \in \mathrm{ca}(\mathcal{E}, \mathcal{B})$ by the formula

$$U_t\mu(A) = \int_{\mathcal{E}} P(t, x, A)\mu(dx),$$

and let (f, μ) denote the action of a linear functional on $\mu \in \mathrm{ca}(\mathcal{E}, \mathcal{B})$. A $\mathcal{B}$-measurable function $f(\cdot)$ on $\mathcal{E}$ is a *moment* iff $f \geqq 0$ and $\inf_{x \in \mathcal{E} - K_n} f(x) \to \infty$ as some sequence of compact $K_n \uparrow \mathcal{E}$.

THEOREM. *The following conditions are equivalent:*

(i) *x_t has a positive finite invariant measure;*

(ii) $\exists \mu \in C^+$, $\exists$ *moment g, with* $\sup_{t \geqq 0}(g, U_t\mu) < \infty$;

(iii) $\exists \mu \in C^+$ *such that the measures $U_t\mu$, $t \geqq 0$, are uniformly countably additive;*

(iv) $\exists \mu, \nu \in C^+$ *such that the measures $U_t\mu$, $t \geqq 0$ are uniformly ν-continuous;*

(v) $\exists \mu \in C^+$, $\exists K$ *weakly compact in* $\mathrm{ca}(\mathcal{E}, \mathcal{B})$, *$K$ contains the orbit* $\{U_t\mu, t \geqq 0\}$.

LEMMA 1. *For K compact,*

$$\lim_{h \downarrow 0} \sup_{y \in K} \left| P(t+h, y, A) - P(t, y, A) \right| = 0.$$

PROOF. Pick $r > 0$ so that $U = S_r(K) = \{y : \rho(y, K) < r\}$ has compact closure. Then $P(h, y, U) \geq P(h, y, S_r(y))$, and $P(h, y, U) \to 1$ as $h \downarrow 0$, uniformly on K. We write for any $\delta > 0$

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$$P(t+h, y, A) - P(t, y, A)$$
$$= \left\{ \int_{U \cap S_\delta(y)} + \int_{\mathcal{E} - (U \cap S_\delta(y))} \right\} [P(t, w, A) - P(t, y, A)] P(h, y, dw).$$

Let $\epsilon > 0$ be given. The closure of U is compact, so it is possible to pick $\delta > 0$ so small that $w, z \in U$, $\rho(w, z) \leqq \delta$ imply $|P(t, w, A) - P(t, z, A)| \leqq \epsilon/2$; with this choice of δ the first integral is at most $\epsilon/2$ in magnitude. The second integral is in magnitude at most

$$1 - P(h, y, S_\delta(y)) - P(h, y, U) + P(h, y, S_\delta(y) \cup U),$$

which tends to 0 as $h \downarrow 0$ uniformly on K.

LEMMA 2. *For $t > 0$, K compact,*

$$\lim_{h \downarrow 0} \sup_{y \in K} | P(t-h, y, A) - P(t, y, A) | = 0.$$

PROOF. Let $\epsilon > 0$ be given. Choose $r > 0$ so that $S_r(K)$ has compact closure and then $\delta < t$ so small that $s \leqq \delta$ implies

$$\sup_{y \in K} P(s, y, \mathcal{E} - S_r(K)) \leqq \epsilon/2.$$

Considering only $0 \leqq h \leqq \delta$, we write

$$P(t-h, y, A) - P(t, y, A) = \left\{ \int_{S_r(K)} = \int_{\mathcal{E} - S_r(K)} \right\}$$
$$\cdot [P(t-\delta, z, A) - P(t-\delta+h, z, A)] P(\delta - h, y, dz).$$

By Lemma 1 choose $s \leqq \delta$ so small that $h \leqq s$ implies

$$\sup_{z \in S_r(K)} | P(t-\delta, z, A) - P(t-\delta+h, z, A) | \leqq \epsilon/2.$$

LEMMA 3. *For fixed $A \in \mathcal{B}$, $P(t, y, A)$ is (t, y)-continuous in the product topology of $[0, \infty) \times \mathcal{E}$.*

PROOF. Let $\epsilon > 0$ be given. Writing, with $h > 0$ if $t = 0$,

$$P(t, y, A) - P(t+h, z, A)$$
$$= P(t, y, A) - P(t, z, A) + P(t, z, A) - P(t+h, z, A),$$

we pick by Lemmas 1 and 2, and by the z-continuity of $P(t, z, A)$, a neighborhood U of y with compact closure and a number $s > 0$ such that $|h| < s$ implies

$$\sup_{z\in U} | P(t, z, A) - P(t+h, z, A)| < \epsilon/2$$

and

$$\sup_{z\in U} | P(t, y, A) - P(t, z, A)| < \epsilon/2.$$

PROOF OF THEOREM. We show (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v)$\Rightarrow$(i).

(i)$\Rightarrow$(ii): $\mathcal{E}$ is σ-compact. If ν is finite and invariant, choose K_n compact $\uparrow \mathcal{E}$. If a_k, $k\geqq 1$ is a nonnegative summable sequence, then there exists a nonnegative sequence b_k, $k\geqq 1$ with $b_k\to\infty$ as $k\to\infty$ and $\sum_{n=1}^\infty a_n b_n<\infty$. Set $a_n=\nu(K_{n+1}-K_n)$, $g(x)=b_n$ for $x\in K_{n+1}-K_n$. Then g is a moment and $(g, U_t\nu)\equiv$constant for $t\geqq 0$.

(ii)$\Rightarrow$(iii): By Lemma 1 and Dini's theorem [**2**, p. 239], $P(t,x,A)\to 0$ as $A\downarrow\phi$, uniformly on (t, x)-compacta. Let $s>0$ be fixed, K_n compact $\uparrow \mathcal{E}$. For $t\leqq s$,

$$U_t\mu(A) \leqq \sup_{y\in K_n} P(t, y, A)\mu(K_n) + \mu(\mathcal{E} - K_n).$$

For $t>s$, with $c=\sup_{t\geqq 0}(g, U_t\mu)$,

$$U_t\mu(A) \leqq \sup_{y\in K_n} P(s, y, A)\mu(\mathcal{E}) + c/\inf_{x\in\mathcal{E}-K_n} g(x).$$

With $\epsilon>0$ given, choose first n so large that

$$\mu(\mathcal{E} - K_n) \leqq c/\inf_{x\in\mathcal{E}-K_n} g(x) < \epsilon/2,$$

and then choose $B\in\mathcal{B}$ so near ϕ that $A\subseteq B$ implies

$$\mu(\mathcal{E}) \sup_{0\leqq h\leqq s} \sup_{y\in K_n} P(h, y, A) < \epsilon/2.$$

Hence $\lim_{A\downarrow\phi}\sup_{t\geqq 0} U_t\mu(A)=0$, and this implies uniform countable additivity.

(iii)$\Rightarrow$(iv)$\Rightarrow$(v): By [**3**, Theorem 1, p. 305], the orbit $\{U_t\mu, t\geqq 0\}$ is weakly sequentially compact. Uniform absolute continuity w.r. to a fixed finite positive measure follows from [**3**, Theorem 2, p. 306]. By the Eberlein-Šmulyan theorem [**3**, p. 430] the weak closure of the orbit is weakly compact.

(v)$\Rightarrow$(i): Each U_t is weakly continuous, since for $f\in \text{ca}(\mathcal{E}, \mathcal{B})^*$, $(f, U_t\cdot)$ is just another continuous linear functional. Let O be the orbit $\{U_t\mu, t\geqq 0\}$, and let co O and clco O be its convex and closed convex hull, respectively (weak topology). Clearly clco $O\subseteq$clco K. Since K is weakly compact, the Krein-Šmulyan theorem implies that clco K is itself weakly compact. Since clco O is a closed subset of cl co K, it too is weakly compact. If now $\nu \in$ cl co O, there exists a generalized sequence $\nu_\alpha \in$ co O with $\nu_\alpha \to \nu$ weakly. Thus $U_t\nu_\alpha \in$ co O, and $U_t\nu_\alpha \to U_t\nu$ weakly. Hence each U_t carries cl co O into itself. Evidently $U_tU_s = U_sU_t$. Thus by the Markov-Kakutani fixed-point theorem [**3**, p. 456], there exists a measure $\nu \in$ cl co O with $U_t\nu = \nu$ for all $t \geqq 0$. Since $\mu > 0$, the orbit O is confined to a sphere bounded away from the origin and ν can be normalized as a probability measure.

To see this, we use a result of Mazur [**3**, p. 422] to the effect that cl co O is closed in the norm topology. Then ν is a strong limit point of elements in co O. There is no restriction in taking $\|\mu\| = 1$ so that co O consists entirely of probability measures. We will be able to find a sequence $\nu_n \in$ co O with $\|\nu_n - \nu\| \to 0$ (sequences are adequate since strong topology is metric). Since $\|\nu_n\| = 1$, it follows that $\|\nu\| = 1$.

S. P. Lloyd has remarked that ν will be unique within cl co O. Let $\zeta \in$ cl co O be invariant. By Mazur's theorem there exist sequences $\nu_n \to \nu$, $\zeta_m \to \zeta$ in norm with $\nu_n, \zeta_m \in$ co O, i.e., $\nu_n = P_n\mu$, $\zeta_m = P_m\mu$ for convex combinations P_n, P_m of operators U_t. Clearly $P_m\nu = \nu$, $P_n\zeta = \zeta$, $P_nP_m = P_mP_n$. Thus

$$\|\nu - \zeta\| = \|P_m\nu - P_n\zeta\| = \|P_m(\nu - P_n\mu) + P_n(P_m\mu - \zeta)\|.$$

References

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