A LOWER BOUND FOR PERMANENTS OF (0, 1)-MATRICES[1]

HENRYK MINC

1. **Introduction.** If $A=(a_{ij})$ is an n-square matrix then the permanent of A is defined by

$$\operatorname{per}(A) = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Clearly the permanent function is invariant under permutations of rows and columns and under matrix transposition. An up-to-date account of the theory of permanents and an extensive bibliography on the subject is to be found in [2]. The permanent function plays an important part in combinatorics. In fact, the permanent of a (0, 1)-matrix (i.e., a matrix all of whose entries are 0 and 1) is the number of systems of distinct representatives for the corresponding configuration [7, p. 54]. It is therefore of considerable interest to determine the bounds for permanents of (0, 1)-matrices with prescribed row sums (and/or column sums). Let r_i and c_j denote the ith row sum and the jth column sum of A respectively, i.e.,

$$r_i = \sum_{j=1}^{n} a_{ij}, \qquad c_j = \sum_{i=1}^{n} a_{ij}.$$

It is obvious that if A is a (0, 1)-matrix then

$$\operatorname{per}(A) \leqq \prod_{i=1}^{n} r_i. \tag{1}$$

In (4) I conjectured that

$$\operatorname{per}(A) \leqq \prod_{i=1}^{n} (r_i!)^{1/r_i} \tag{2}$$

and proved that

$$\operatorname{per}(A) \leqq \prod_{i=1}^{n} \frac{r_i+1}{2} \tag{3}$$

and that equality holds in (3) if and only if A is a permutation matrix.

Presented to the Society, January 25, 1967; received by the editors September 20, 1966.

[1] This work was supported by the Air Force Office of Scientific Research under Grant AFOSR 432-63.

Jurkat and Ryser [1] improved (3):

$$\text{per}(A) \leqq \prod_{i=1}^{n} (r_i!)^{1/n}\left(\frac{r_i+1}{2}\right)^{(n-r_i)/n}. \tag{4}$$

In a recent paper [6] I proved that

$$\text{per}(A) \leqq \prod_{i=1}^{n} \frac{r_i+\sqrt{2}}{1+\sqrt{2}} \tag{5}$$

and that equality holds in (5) if and only if there exist permutation matrices P and Q such that PAQ is a direct sum of 1-square and 2-square matrices all of whose entries are 1. The bounds in (4) and in (5) are not comparable.

Clearly a permanent of a (0, 1)-matrix is always nonnegative. In the case of (0, 1)-matrices all of whose row sums and column sums are equal to k, it is conjectured [7, pp. 59, 77] that

$$\text{per}(A) \geqq n!(k/n)^n, \tag{6}$$

with equality if and only if $k=n$. The inequality is known to be true in a very special case, when A is positive semidefinite. It is also easy to see that if every row sum and column sum of A is equal to k then [3, p. 61]

$$\text{per}(A) \geqq k. \tag{7}$$

In [1] Jurkat and Ryser used a remarkable, but involved, method to show that if A is any n-square (0, 1)-matrix then

$$\text{per}(A) \geqq \prod_{i=1}^{n} \{r_i + 1 - i\} \tag{8}$$

where $\{r_i+1-i\} = \max(r_i+1-i,\ 0)$. In the present paper, an inequality essentially equivalent to (8) is proved by a substantially simpler method and the case of equality is discussed.

2. **Results.** Let A be an n-square (0, 1)-matrix. Assume that $r_1 \geqq \cdots \geqq r_n$. Let $\overline{A}$ be the *maximal matrix* [7] with row sums $r_1, \cdots, r_n$ (i.e., the first r_i entries in the ith row of $\overline{A}$ are 1 and the other entries are 0, $i=1, \cdots, n$).

THEOREM. *If A is an n-square* (0, 1)*-matrix with row sums $r_1 \geqq \cdots \geqq r_n$ and $\overline{A}$ the maximal matrix with the same row sums, then*

$$\text{per}(A) \geqq \prod_{i=1}^{n} \{r_i + i - n\} \tag{9}$$

where $\{r_i+i-n\}=r_i+i-n$ *if* $r_i+i-n\geqq 0$ *and* $\{r_i+i-n\}=0$ *otherwise. If* $\operatorname{per}(A)\neq 0$, *then equality holds in* (8) *if and only if* $AP=\bar{A}$ *for some permutation matrix* P.

PROOF. If $r_n=0$ there is nothing to prove. Otherwise, we can assume, without loss of generality, that $a_{nj}=1, j=1, \cdots, r_n$. Expanding the permanent by the last row [2, p. 578], we have

$$\operatorname{per}(A) = \sum_{j=1}^{r_n} \operatorname{per}(A(n\,|\,j))$$

where $A(i|j)$ denotes the $(n-1)$-square submatrix of A obtained by deleting the ith row and the jth column of A. Now use induction on n. The case $n=1$ is trivial. Assume that the theorem holds for all $(n-1)$-square matrices and thus

$$\begin{aligned}\operatorname{per}(A(n\,|\,j)) &\geqq \prod_{i=1}^{n-1} \{(r_i - a_{ij}) + i - (n-1)\} \\ &\geqq \prod_{i=1}^{n-1} \{r_i + i - n\}, \qquad j = 1, \cdots, r_n,\end{aligned} \tag{10}$$

since $1-a_{ij}\geqq 0$. Hence

$$\begin{aligned}\operatorname{per}(A) &\geqq \sum_{j=1}^{r_n}\prod_{i=1}^{n-1} \{r_i + i - n\} \\ &= r_n \prod_{i=1}^{n-1} \{r_i + i - n\} \\ &= \prod_{i=1}^{n} \{r_i + i - n\}.\end{aligned}$$

Now, consider the case of equality. We assert that if

$$\operatorname{per}(A) = \prod_{i=1}^{n} (r_i + i - n) > 0 \tag{11}$$

then A must be the maximal matrix $\bar{A}$, possibly with permuted columns. We use induction on n. Assume that the columns of A have been permuted, if necessary, so that the first r_n entries in the last row are 1 and the other entries are 0. Clearly (11) implies that $r_n>0$ and that equality must hold in (10) for $j=1, \cdots, r_n$. Therefore $a_{ij}=1, i=1, \cdots, n-1, j=1, \cdots, r_n$. In other words, all the entries in the first r_n columns of A are 1. It follows that $A(n|1)= \cdots =A(n|r_n)$. Therefore

$$\begin{aligned} 0 < \operatorname{per}(A) &= \sum_{j=1}^{r_n} \operatorname{per}(A(n \mid j)) \\ &= r_n \operatorname{per}(A(n \mid 1)). \end{aligned}$$

On the other hand, by (11),

$$\begin{aligned} \operatorname{per}(A) &= \prod_{i=1}^{n} (r_i + i - n) \\ &= r_n \prod_{i=1}^{n-1} (r_i + i - n). \end{aligned}$$

Hence

$$\operatorname{per}(A(n \mid 1)) = \prod_{i=1}^{n-1} (r_i + i - n)$$

and, by the induction hypothesis, the matrix $A(n|1)$ is a maximal matrix possibly with its last $n-r_n$ columns permuted. But $a_{i1}=1$, $i=1, \cdots, n$, and $a_{nj}=1$, $j=1, \cdots, r_n$. Thus A is the maximal matrix $\overline{A}$ modulo a permutation of columns.

We prove the converse by showing that

$$\operatorname{per}(\overline{A}) = \prod_{i=1}^{n} \{r_i + i - n\}. \tag{12}$$

If $r_n=0$ then both sides of (12) are 0. Assume therefore that $r_n>0$, expand the permanent by the last row and use induction on n. Thus

$$\operatorname{per}(\overline{A}) = \sum_{j=1}^{r_n} \operatorname{per}(\overline{A}(n \mid j)). \tag{13}$$

Clearly

$$\overline{A}(n \mid 1) = \cdots = \overline{A}(n \mid r_n). \tag{14}$$

Also, $\overline{A}(n|1)$ is a maximal matrix with row sums $r_1-1, r_2-1, \cdots, r_{n-1}-1$. Thus, by the induction hypothesis,

$$\begin{aligned} \operatorname{per}(\overline{A}(n \mid 1)) &= \prod_{i=1}^{n-1} \{(r_i - 1) + i - (n - 1)\} \\ &= \prod_{i=1}^{n-1} \{r_i + i - n\}. \end{aligned}$$

Hence, by (13) and (14),

$$\begin{aligned}\operatorname{per}(\overline{A}) &= r_n \prod_{i=1}^{n-1} \{r_i + i - n\} \\ &= \prod_{i=1}^{n} (r_i + i - n).\end{aligned}$$

COROLLARY 1. *If A is an n-square* (0, 1)*-matrix with row sums $r_1 \geqq \cdots \geqq r_n$ and column sums $c_1 \geqq \cdots \geqq c_n$ then*

$$\operatorname{per}(A) \geqq \max\left(\prod_{i=1}^{n} \{r_i + i - n\},\ \prod_{j=1}^{n} \{c_j + j - n\} \right). \tag{15}$$

Equality occurs in (15) *if and only if either $A = \overline{A}$ or* $\operatorname{per}(A) = 0$.

COROLLARY 2. *If A is an n-square* (0, 1)*-matrix with row sums $r_1 \geqq \cdots \geqq r_n$ and* $\operatorname{per}(A) = 0$ *then there exists an integer s, $1 \leqq s \leqq n$, such that the last s rows of A contain at least $s(n-s+1)$ zeros.*

For, if $\operatorname{per}(A) = 0$ then the theorem implies that $r_t + t - n \leqq 0$ for some t, i.e., $r_t \leqq n - t$ and therefore $r_i \leqq n - t$, $i = t, \cdots, n$. But the number of zeros in the ith row is $n - r_i$. Set $s = n - t + 1$. Then the number of zeros in the last s rows is

$$\begin{aligned}\sum_{i=t}^{n} (n - r_i) &\geqq \sum_{i=t}^{n} n - (n - t) \\ &= (n - t + 1)t \\ &= s(n - s + 1).\end{aligned}$$

Corollary 2 is, of course, an immediate consequence of the well-known Frobenius-König theorem [3, p. 51].

REFERENCES

1. W. B. Jurkat and H. J. Ryser, *Matrix factorizations of determinants and permanents*, J. Algebra **3** (1966), 1–27.

2. Marvin Marcus and Henryk Minc, *Permanents*, Amer. Math. Monthly **72** (1965), 577–591.

3. ———, *Modern university algebra*, Macmillan, New York, 1966.

4. Henryk Minc, *Upper bounds for permanents of* (0, 1)*-matrices*, Bull. Amer. Math. Soc. **69** (1963), 789–791.

5. ———, *Permanents of* (0, 1)*-circulants*, Canad. Math. Bull. **7** (1964), 253–263.

6. ———, *An inequality for permanents of* (0, 1)*-matrices*, J. Combinatorial Sci. **2** (1967), 321–326.

7. H. J. Ryser, *Combinatorial mathematics*, Carus Math. Monograph No. 14, 1963.

UNIVERSITY OF CALIFORNIA, SANTA BARBARA