GROUPS OF ELLIPTIC LINEAR FRACTIONAL TRANSFORMATIONS

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1. **Introduction.** Our aim is to give a simple and self contained statement and proof of the following theorem:

Every subgroup of the group of linear fractional transformations of the complex plane, which contains only elliptic transformations, is conjugate to the image under stereographic projection of a group of rotations of a sphere.

An important consequence is that every discontinuous (or discrete) elliptic group is finite.

Simple proofs of these well-known results do not seem to be easily accessible. Indeed, we are indebted to J. Lehner for the only reference we know, to Fatou [**1**], where an argument similar to that given here is presented in a much fuller context.

2. **Definitions and preliminaries.** A *linear fractional transformation* is a map $T: z \to (az+b)/(cz+d)$, of the (extended) complex plane onto itself, with complex coefficients and nonvanishing determinant. We may require $ad-bc=1$; then the matrix is determined up to a factor -1, and we may write

$$T = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

These transformations form a group $\mathfrak{L}$, with composition corresponding to matrix multiplication.

An *elliptic transformation* is one that is conjugate to a rotation $z \to kz$, $|k| = 1$, of the complex plane. More explicitly, T is elliptic if $T = I$, the identity map $z \to z$, or if it has distinct eigenvalues λ and $\lambda^{-1} = \bar{\lambda}$ (complex conjugate) with $|\lambda| = 1$. Then T has trace $a+d = \lambda + \bar{\lambda}$ real, with $|a+d| \leqq 2$. It follows also that T has two distinct fixed points.

Consider the finite complex plane as a plane in three dimensional space with coordinate axes (x, y, t) forming a right handed system. Let Σ be the unit sphere, with equator the circle $|z| = 1$ and north pole N. *Stereographic projection* carries each point $P \neq N$ on Σ into the point P' where the line NP meets the plane, with $N' = \infty$. If $\tilde{T}$ is a rotation of Σ; there is a unique transformation T in $\mathfrak{L}$ such that, when $\tilde{T}(P) = Q$, $T(P') = Q'$. Let $\mathfrak{R}$ be the group of all linear fractional

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transformations T obtained in this way from a rotation $\tilde{T}$ of Σ.

Let $\tilde{A}_\theta$ denote the rotation of Σ around the t axis which turns the positive x axis by an angle θ towards the positive y axis. The corresponding element A_θ in $\mathcal{R}$ is

$$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}.$$

Let $\tilde{B}_\phi$ denote the rotation of Σ around the y axis which turns the positive x axis by an angle ϕ towards the positive t axis. The corresponding element B_ϕ in $\mathcal{R}$ is

$$\begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}.$$

A general rotation $\tilde{T}$ of Σ will have a fixed point P whose spherical coordinates are (ϕ, θ), and a rotation angle α. If O is the origin, ϕ is the angle between ON and OP satisfying $0 \leqq \phi \leqq \pi$, while θ is the angle between the positive x axis and the projection of OP in the xy plane measured counterclockwise and satisfying $0 \leqq \theta < 2\pi$. We take $0 \leqq \alpha < 2\pi$ as the clockwise angle of rotation as viewed from the center of the sphere. Now T can be written as the composition $(\tilde{C})^{-1}\tilde{A}_\alpha\tilde{C}$ where $\tilde{C} = \tilde{B}_{-\phi}\tilde{A}_{-\theta}$ takes P to N. The composition of the corresponding elements of L yields

$$\begin{pmatrix} \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\cos\phi & i\sin\frac{\alpha}{2}e^{i\theta}\sin\phi \\ i\sin\frac{\alpha}{2}e^{-i\theta}\sin\phi & \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\cos\phi \end{pmatrix}.$$

We note that T in $\mathcal{R}$ has the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

with $|a|^2 + |b|^2 = 1$. Conversely, if an element T of $\mathcal{L}$ is of the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

with $|a|^2 + |b|^2 = 1$, and is not the identity, then the equations

(a) $$\cos\frac{\alpha}{2} = \operatorname{Re} a \qquad (0 \leqq \alpha < 2\pi)$$

(b) $$\theta = \arg b - \frac{\pi}{2} \qquad (0 \leqq \theta < 2\pi)$$

(c) $$\sin\phi = \frac{|b|}{\sqrt{(1-(\operatorname{Re} a)^2)}}, \qquad \cos\phi = \frac{\operatorname{Im} a}{\sqrt{(1-(\operatorname{Re} a)^2)}} \qquad (0 \leqq \phi \leqq \pi)$$

admit unique solutions in the indicated ranges. These numbers are the parameters of a rotation $\tilde{T}$ whose corresponding element in $\mathfrak{L}$ is T and hence T is in $\mathfrak{R}$.

We have shown that T is in $\mathfrak{R}$ if and only if it has the form

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1,$$

by a method which yields at the same time the parameters of the corresponding rotation $\tilde{T}$.

3. **The theorem.** If $\mathcal{G} = (T_0, T_1, \cdots)$ is any subgroup of $\mathfrak{L}$, and U any element of $\mathfrak{L}$, then $\mathcal{G}' = (UT_0U^{-1}, UT_1U^{-1}, \cdots)$ is another subgroup, isomorphic to $\mathcal{G}$. (Indeed, we may think of $\mathcal{G}'$ as obtained from $\mathcal{G}$ by the (conformal) change of coordinates $z \to u(z)$.) Any group $U\mathcal{G}U^{-1}$ is a *conjugate* of the group $\mathcal{G}$.

THEOREM. *Let $\mathcal{G}$ be a subgroup of the linear fractional group $\mathfrak{L}$ such that $\mathcal{G}$ contains only elliptic transformations. Then some group $\mathcal{G}' = U\mathcal{G}U^{-1}$, conjugate to $\mathcal{G}$ by an element U of $\mathfrak{L}$, is contained in the subgroup $\mathfrak{R}$ of $\mathfrak{L}$, induced by rotations of the unit sphere.*

4. **The proof.** Let there be given a group $\mathcal{G}$ of elliptic transformations. If $\mathcal{G}$ contains only the identity transformation

$$I = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

the assertion becomes trivial. Assume that $\mathcal{G}$ contains some $T \neq I$. Then the quadratic equation $z = T(z)$ has distinct solutions z_1 and z_2, the fixed points of T. Now the conditions $U(0) = z_1$ and $U(\infty) = z_2$ have a solution U in $\mathfrak{L}$ (in fact we still have left one degree of freedom in the choice of U). Therefore the element $U^{-1}TU$ of $U^{-1}\mathcal{G}U$ has fixed points 0 and ∞. After replacing $\mathcal{G}$ by its conjugate $U^{-1}\mathcal{G}U$ we may assume that $\mathcal{G}$ contains some $T \neq I$ with fixed points 0 and ∞.

Let $\mathcal{G}_0$ be the subgroup of $\mathcal{G}$ consisting of all T in $\mathcal{G}$ with 0 and ∞ as fixed points. The equations $T(0) = 0$ and $T(\infty) = \infty$ imply T is diagonal,

$$T = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

The condition $ad-bc=1$ gives $d=a^{-1}$. Since T is elliptic, $|a|=1$, whence $d=\bar{a}$. Thus the elements of $\mathcal{G}_0$, of the form

$$\begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix},$$

with $|a|=1$, belong to the group $\mathcal{R}$ of all T of the form

$$T = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

We show next that every T in $\mathcal{G}$ has the form

$$T = \begin{pmatrix} a & b \\ c & \bar{a} \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and choose

$$S = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

in $\mathcal{G}_0$, with $S \neq I$, hence with $\lambda \neq \pm 1$. Now

$$ST = \begin{pmatrix} \lambda a & \lambda b \\ \bar{\lambda} c & \bar{\lambda} d \end{pmatrix},$$

and the two elliptic elements T and ST of $\mathcal{G}$ must have real traces, $a+d$ and $\lambda a+\bar{\lambda} d$. It follows first that $d=\bar{a}+h$, h real, next that $\bar{\lambda} h$ is real, whence $h=0$ and $d=\bar{a}$.

We show that, for T in $\mathcal{G}$, $b=0$ iff $c=0$. Suppose $c=0$, hence

$$T = \begin{pmatrix} a & b \\ 0 & \bar{a} \end{pmatrix},$$

and take S as before. Computation shows that, since $a\bar{a}=1$,

$$STS^{-1} = \begin{pmatrix} a & \lambda^2 b \\ 0 & \bar{a} \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} \bar{a} & -b \\ 0 & a \end{pmatrix}.$$

Then

$$K = STS^{-1}T^{-1} = \begin{pmatrix} 1 & (\lambda^2 - 1)b \\ 0 & 1 \end{pmatrix}.$$

Since K, in $\mathcal{G}$, is elliptic and has eigenvalue 1, we must have $K=I$, that is, $(\lambda^2-1)b=0$, and therefore $b=0$. The proof that $b=0$ implies $c=0$ is similar.

We now show that there exists a real number $r\neq 0$ such that every T in $\mathcal{G}$ has the form

$$\begin{pmatrix} a & b \\ r\bar{b} & \bar{a} \end{pmatrix}.$$

For this it suffices to show that if $\mathcal{G}$ contains elements

$$T_i = \begin{pmatrix} a_i & b_i \\ r_i\bar{b}_i & \bar{a}_i \end{pmatrix}, \qquad i = 1, 2,$$

with $b_i\neq 0$, $i=1, 2$, then r_1 and r_2 are real and equal. Reality of r_i follows from the condition that $ad-bc=a\bar{a}-rb\bar{b}=1$. We compute the diagonal elements of the product:

$$T_1T_2 = \begin{pmatrix} a_1a_2 + r_2b_1\bar{b}_2 & * \\ * & r_1\bar{b}_1b_2 + \bar{a}_1\bar{a}_2 \end{pmatrix}.$$

These elements must be complex conjugates, whence $r_2b_1\bar{b}_2$ is conjugate to $r_1\bar{b}_1b_2$, $r_2b_1\bar{b}_2=r_1b_1\bar{b}_2$, and $r_2=r_1$.

We show that we may choose $r=\pm 1$. We have that every T in $\mathcal{G}$ has the form

$$\begin{pmatrix} a & b \\ r\bar{b} & \bar{a} \end{pmatrix}$$

with real $r\neq 0$. Conjugating $\mathcal{G}$ by

$$V = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix},$$

in $\mathcal{L}$, carries each T into

$$VTV^{-1} = \begin{pmatrix} a & v^2b \\ v^{-2}r\bar{b} & \bar{a} \end{pmatrix}$$

with

$$r' = (v^{-2}r\bar{b})/(\overline{v^2b}) = r/(v\bar{v})^2.$$

Choosing v such that $|v|^4=|r|$ gives $r'=\pm 1$. (Here we have exploited our remaining freedom in the choice of U.)

We show that $r=1$ is impossible unless each T in $\mathcal{G}$ has $b=c=0$. Suppose then that $r=+1$ and $\mathcal{G}$ contains

$$T=\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

with $b\neq 0$. If

$$S=\begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \qquad \lambda \neq \pm 1,$$

then

$$STS^{-1}=\begin{pmatrix} a & \lambda^2 b \\ \bar{\lambda}^2\bar{b} & \bar{a} \end{pmatrix},$$

$$T^{-1}=\begin{pmatrix} \bar{a} & -b \\ -\bar{b} & a \end{pmatrix},$$

and

$$K=STS^{-1}T^{-1}=\begin{pmatrix} a\bar{a}-\lambda^2 b\bar{b} & * \\ * & * \end{pmatrix}.$$

Now the trace of K is twice the real part of $a\bar{a}-\lambda^2 b\bar{b}$, that is, twice $a\bar{a}+tb\bar{b}$, where t is the real part of $-\lambda^2$, whence $a\bar{a}+tb\bar{b}\leqq 1$. But this is impossible, since T has determinant $a\bar{a}-b\bar{b}=1$, while $-1<t<+1$.

We have shown that we may always take $r=-1$. Thus G consists solely of transformations of the form

$$T=\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},$$

and $\mathcal{G}$ is a subgroup of $\mathcal{R}$, as required.

Reference

1. P. Fatou, *Fonctions automorphes*, Volume II of *Théorie des fonctions algébriques*, by P. Appel and E. Goursat, Gauthier-Villars, Paris, 1930.

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