SOME IDENTITIES RELATED TO PÓLYA'S PROPERTY W FOR LINEAR DIFFERENTIAL EQUATIONS

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1. **Introduction.** In this note we study relationships between matrix solutions of

$$X' = AX \tag{1}$$

and

$$X' = BX \tag{2}$$

where $B = -T^{-1}A^*T$ for some constant matrix T satisfying $T^*T^{-1} = I$ or $T^*T^{-1} = -I$. A familiar example is the case when (1) is the classical vector matrix representation of

$$Ly = y^k + \sum_{i=0}^{k-2} a_i y^i = 0. \tag{3}$$

Then, for $T = ((-1)^i \delta_{i,k+1-j})$, (2) will represent the adjoint equation

$$L^+ y = (-1)^k y^k + \sum_{i=0}^{k-2} (-1)^i (\bar{a}_i y)^i = 0. \tag{4}$$

An application of some of our results to this case will yield that certain sets of fundamental solutions of (3) have property W—for a definition see [**3**]—if and only if certain sets of solutions of (4) have it. We will obtain identities among minors of Wronskians associated with (3) and (4).

2. **Determinantal identities.** Let $A = (a_{ij})$ be a $k \times k$ matrix of continuous complex valued functions on some interval such that tr $A = 0$. Let T be a $k \times k$ constant matrix such that $T^*T^{-1} = I$ or $T^*T^{-1} = -I$ and let $B = -T^{-1}A^*T$. For a given u let $M(t, u)$ and $N(t, u)$ denote the unique matrix solutions of $X' = AX$ with $X(u) = I$ and $Y' = BY$ with $Y(u) = I$, respectively.

Lemma 1. $M(t, u)M(u, v) = M(t, v)$.

Proof. This is immediate from the fact that for any nonsingular solution ϕ of $X' = AX$, $M(t, u) = \phi(t)\phi^{-1}(u)$, which follows from the

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uniqueness of the solution to the system $X'=AX$ with $X(u)=I$. Of course, the same result is valid for N.

LEMMA 2. tr $M(t, u)=1=$ tr $N(t, u)$.

PROOF. This follows from the fact that if $X'=AX$, then $(\det X)' = (\operatorname{tr} A)(\det X)$—see [1, Theorem 7.3, p. 28].

THEOREM 1. $M(t, u)=T^{-1}N^*(u, t)T$.

PROOF. For fixed u let $X(t)=T^{-1*}M^*(t, u)T^*N(t, u)$. Then a simple computation yields that $X'(t)=0$ and $X(u)=I$. Hence $X(t)=I$. This is equivalent to the theorem in view of $N(u, t)N(t, u)=I$ which follows from Lemma 1.

Let $P(t, u)=(P_{ij}(t, u))$ where $P_{ij}(t, u)$ is $(-1)^{i+j}$ times the minor of $N_{ij}(t, u)$ in $N(t, u)$. Then $N(u, t)=N(t, u)^{-1}$ and $\det N(t, u)=1$ imply that $N(u, t)=\overline{P}^*(t, u)$. Combining this with Theorem 1 we obtain

THEOREM 2. $M(t, u)=T^{-1}\overline{P}(t, u)T$.

This theorem can be made into a much stronger appearing result by using a known fact on adjugate determinants: If $C=(c_{ij})$ is a $k\times k$ matrix and $D=(d_{ij})$ where d_{ij} is the cofactor of c_{ij}, then any algebraic minor of order r, $1\leqq r<k$ of D is equal to its algebraic complement in C times $(\det C)^{r-1}$. Combining the above with Theorem 2 we obtain

THEOREM 3. *Any $r\times r$ $1\leqq r<k$ algebraic minor of $\overline{T}\overline{N}(t, u)\overline{T}^{-1}$ is equal to its algebraic complement in $M(t, u)$.*

3. **Subwronskians relative to L and L^+.** We now wish to apply some of the above identities to subwronskians of solutions of $Ly=0$ and $L^+y=0$. This is accomplished by specializing A to

$$A = \begin{bmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & 1 \\ -a_0, & -a_1, & \cdots, & & -a_{k-2}, & 0 \end{bmatrix},$$

and letting $T=((-1)^i\delta_{i,k+1-j})$. Now (1) represents (3) and (2) represents (4). Let y_i, z_j $j=1, \cdots, k$ be solutions of (3) and (4), respectively, such that for some number a

$$\overset{i}{y}_j(a) = 1 \quad \text{if } i = j-1,$$
$$\phantom{\overset{i}{y}_j(a)} = 0 \quad \text{if } i \neq j-1,$$

and

$$\overset{i}{z}_j(a) = 1 \quad \text{if } i = j-1,$$
$$\phantom{\overset{i}{z}_j(a)} = 0 \quad \text{if } i \neq j-1.$$

An application of Theorem 3 yields:

COROLLARY 1. *Let* $\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_j\}$ *and* $\beta = \{\beta_1, \beta_2, \cdots, \beta_j\}$ *be increasing subsequences of* $1, 2, 3, \cdots, k$ *and let*

$$\alpha' = \{k+1-\alpha_j, k+1-\alpha_{j-1}, \cdots, k+1-\alpha_1\}$$

and

$$\beta' = \{k+1-\beta_j, k+1-\beta_{j-1}, \cdots, k+1-\beta_1\}.$$

Then—using the notation of [2]—$d(M(t, u)[\alpha|\beta]) = d(\overline{N}(t, u)(\alpha'|\beta'))$.

Among these identities the following ones are of particular interest in connection with Pólya's property W. In the notation of [3],

$$W(y_1, \cdots, y_j) = W(\bar{z}_1, \cdots, \bar{z}_{k-j}).$$

It is hoped that these identities have applications to other areas, such as, the study of conjugate points, boundary value problems, etc.

REFERENCES

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