

SOME IDENTITIES RELATED TO PÓLYA'S PROPERTY W FOR LINEAR DIFFERENTIAL EQUATIONS

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1. **Introduction.** In this note we study relationships between matrix solutions of

$$(1) \quad X' = AX$$

and

$$(2) \quad X' = BX$$

where $B = -T^{-1}A^*T$ for some constant matrix T satisfying $T^*T^{-1} = I$ or $T^*T^{-1} = -I$. A familiar example is the case when (1) is the classical vector matrix representation of

$$(3) \quad Ly = y^k + \sum_{i=0}^{k-2} a_i y^i = 0.$$

Then, for $T = ((-1)^i \delta_{i, k+1-j})$, (2) will represent the adjoint equation

$$(4) \quad L^+y = (-1)^k y^k + \sum_{i=0}^{k-2} (-1)^i (\bar{a}_i y)^i = 0.$$

An application of some of our results to this case will yield that certain sets of fundamental solutions of (3) have property W —for a definition see [3]—if and only if certain sets of solutions of (4) have it. We will obtain identities among minors of Wronskians associated with (3) and (4).

2. **Determinantal identities.** Let $A = (a_{ij})$ be a $k \times k$ matrix of continuous complex valued functions on some interval such that $\text{tr } A = 0$. Let T be a $k \times k$ constant matrix such that $T^*T^{-1} = I$ or $T^*T^{-1} = -I$ and let $B = -T^{-1}A^*T$. For a given u let $M(t, u)$ and $N(t, u)$ denote the unique matrix solutions of $X' = AX$ with $X(u) = I$ and $Y' = BY$ with $Y(u) = I$, respectively.

LEMMA 1. $M(t, u)M(u, v) = M(t, v)$.

PROOF. This is immediate from the fact that for any nonsingular solution ϕ of $X' = AX$, $M(t, u) = \phi(t)\phi^{-1}(u)$, which follows from the

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$$\begin{aligned} y_j^i(a) &= 1 \quad \text{if } i = j - 1, \\ &= 0 \quad \text{if } i \neq j - 1, \end{aligned}$$

and

$$\begin{aligned} z_j^i(a) &= 1 \quad \text{if } i = j - 1, \\ &= 0 \quad \text{if } i \neq j - 1. \end{aligned}$$

An application of Theorem 3 yields:

COROLLARY 1. *Let $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_j\}$ and $\beta = \{\beta_1, \beta_2, \dots, \beta_j\}$ be increasing subsequences of $1, 2, 3, \dots, k$ and let*

$$\alpha' = \{k + 1 - \alpha_j, k + 1 - \alpha_{j-1}, \dots, k + 1 - \alpha_1\}$$

and

$$\beta' = \{k + 1 - \beta_j, k + 1 - \beta_{j-1}, \dots, k + 1 - \beta_1\}.$$

Then—using the notation of [2]— $d(M(t, u) [\alpha | \beta]) = d(\bar{N}(t, u) (\alpha' | \beta'))$.

Among these identities the following ones are of particular interest in connection with Pólya's property W . In the notation of [3],

$$W(y_1, \dots, y_j) = W(\bar{z}_1, \dots, \bar{z}_{k-j}).$$

It is hoped that these identities have applications to other areas, such as, the study of conjugate points, boundary value problems, etc.

REFERENCES

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