POINTS OF MINIMUM NORM ON SMOOTH SURFACES IN BANACH SPACES

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THEOREM. *Suppose E is a real Banach space, and ϕ is a continuously Fréchet differentiable real valued function defined on E. Assume that for some c in R there is a u_0 in $\phi^{-1}(c)$ such that $\|u_0\| \leqq \|u\|$ for all u in $\phi^{-1}(c)$. Then* $|\phi'(u_0)\cdot u_0| = \|\phi'(u_0)\| \, \|u_0\|$.

PROOF. We may assume that u_0 and $\phi'(u_0)$ are not zero. Let $K = \text{Ker } \phi'(u_0)$. It will first be shown that $k \in K$ implies $\|u_0+k\| \geqq \|u_0\|$.

Choose $u_1 \in E$ with $\phi'(u_0)\cdot u_1 = 1$ and let $E_2 = \text{span}\{u_1\}$. Then $E = K \times E_2$ and any $u \in E$ can be written $u = (y, \alpha u_1)$ where $y \in K$ and $\alpha = \phi'(u_0)\cdot u$. Let $u_0 = (y_0, \alpha_0 u_1)$. Since $\phi'(u_0) \neq 0$ we may apply the implicit function theorem (see [2]) to obtain a C^1-function $g: U_1 \to R$ where U_1 is a convex open neighborhood of zero in K and g satisfies $g(0) = 0$, $g'(0) = 0$ and

$$\phi(y_0 + h, \alpha_0 u_1 + g(h) u_1) = c \qquad \text{for all } h \in U_1.$$

Let $B = \{u \in E: \|u\| < \|u_0\|\}$. By assumption $\phi^{-1}(c) \cap B = \emptyset$. Assuming $(u_0 + K) \cap B \neq \emptyset$ we will obtain a contradiction. Suppose there is a $k \in K$ with $\|u_0 + k\| < \|u_0\|$. We may assume that $k \in U_1$ and $g(tk) > 0$ for $0 < t \leqq 1$. Then for some s with $0 < s < 1$ we have that $(y_0 + k, \alpha_0 u_1 + sg(k)) \in B$. Since B is convex $(y_0 + \sigma k, \alpha_0 u_1 + \sigma sg(k)) \in B$ for $0 < \sigma \leqq 1$ so $g(\sigma k) \geqq \sigma sg(k)$ for $0 < \sigma \leqq 1$. Therefore $(g(\sigma k) - g(0))/\sigma \geqq \sigma sg(h)/\sigma = sg(k) \neq 0$. In other words $g'(0)\cdot k \neq 0$ and this is a contradiction.

Therefore $\|u_0 + k\| \geqq \|u_0\|$ for all $k \in K$, and it follows that $u_0 \notin K$ and $\|\alpha u_0 + k\| \geqq \|\alpha u_0\|$ for all $\alpha \in R$ and $k \in K$. Let $\epsilon > 0$ and choose $v \in E$ with $\|v\| = 1$ and $|\phi'(u_0)\cdot v| \geqq \|\phi'(u_0)\| - \epsilon$. Then $v = \alpha u_0 + k$, $k \in K$ and we have that $1 = \|\alpha u_0 + k\| \geqq |\alpha| \, \|u_0\|$. Hence $\|\phi'(u_0)\| - \epsilon \leqq |\phi'(u_0)\cdot v| = |\phi'(u_0)\cdot \alpha u_0| \leqq |\phi'(u_0)\cdot u_0| / \|u_0\|$, so $\|\phi'(u_0)\| \, \|u_0\| - \epsilon \|u_0\| \leqq |\phi'(u_0)\cdot u_0|$. Since ϵ is arbitrary $\|\phi'(u_0)\| \, \|u_0\| \leqq |\phi'(u_0)\cdot u_0|$. The reverse inequality is trivial, so the proof is complete.

COROLLARY 1. *Let ϕ, E, and u_0 be as in the theorem. Suppose $E = F^*$ where F is separable and $\phi(u_n) \to \phi(u_0)$ whenever $u_n \cdot x \to u_0 \cdot x$ for all $x \in E$. Then in addition to the above conclusion it follows that $\phi'(u_0) \in F$.*

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PROOF. $\phi'(u_0)\in F^{**}$ and to show that $\phi'(u_0)\in F$ it is sufficient to show that $\phi'(u_0)\cdot u_n\to 0$ whenever $u_n\cdot x\to 0$ for all $x\in F$. Assuming this is false we can find a sequence $\{v_n\}\subset F^*$ with $\|v_n\|=1$ and $|\phi'(u_0)\cdot v_n|>\gamma>0$ for all n and some $\gamma>0$. Let

$$\alpha_n = \max\{\,|\,\phi(u_0 + tv_n) - \phi(u_0)\,| : 0 \leqq t \leqq 1\}.$$

Then $\alpha_n\to 0$ and we have that $|(\phi(u_0+\beta_n v_n)-\phi(u_0))/\beta_n|\leqq\alpha_n/\beta_n\to 0$ where $\beta_n=\alpha_n^{1/2}$. However

$$|\,\phi(u_0 + v) - \phi(u_0) - \phi'(u_0)\cdot v\,| /\|v\| \to 0 \qquad \text{as } \|v\| \to 0$$

so that $|((\phi(u_0+\beta_n v_n)-\phi(u_0))/\beta_n)-(\phi'(u_0)\cdot v_n)|\to 0$ as $n\to\infty$ giving that $|\phi'(u_0)\cdot v_n|\to 0$ which is a contradiction.

COROLLARY 2. *Assume the hypothesis of the above corollary and suppose that* $F=L^1[0, 1]$ *so* $F^*=L^\infty[0, 1]$. *Then* $u_0\in L^\infty$, $\phi'(u_0)\in L^1$ *and we have that* $u_0(t)=\pm\|u_0\|$ sgn $\phi'(u_0)(t)$ *almost everywhere where* $\phi'(u_0)(t)\neq 0$. *In particular, if* $\phi'(u_0)(t)\neq 0$ *almost everywhere,* u_0 *is a "bang-bang" type solution which often occurs in control theory.*

REMARKS. (a) The conclusion above may be phrased in another way. Namely, $\phi'(u_0)/\|\phi'(u_0)\|$ is a support functional to the unit sphere in E at the point $\pm u_0\|u_0\|$. If the norm $N(u)=\|u\|$ is differentiable (except at zero) then support functionals are unique and it follows that $N'(u_0)=\pm\phi'(u_0)$. In this case the above theorem reduces to the Lagrange method of multiplier result.

(b) Corollary 2 is proved in [3, pp. 302–311], for a special class of constraints ϕ. The result there suggested the above theorem. In [3] the argument is basically the following: First extend ϕ to $L^p[0, 1]$, $(1<p<\infty)$. Then, since the norm in L^p is differentiable, a Lagrange multiplier argument applies to give a solution u_p. u_0 is obtained by letting $p\to\infty$.

The norm in $L^\infty[0, 1]$ is nowhere differentiable, and in fact cannot be approximated by a differentiable function [1]. Therefore in using a Lagrange multiplier argument in [3], the indirect approach via L^p was essential.

BIBLIOGRAPHY

1. R. Bonic and J. Frampton, *Differentiable functions on certain Banach spaces*, Bull. Amer. Math. Soc. **71** (1965), 393–395.

2. S. Lang, *Introduction to differentiable manifolds*, Interscience, New York, 1962.

3. T. Saaty and J. Bram, *Nonlinear mathematics*, McGraw-Hill New York, 1964.

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