ON MINIMAL SEPARATING COLLECTIONS

BEZALEL PELEG

1. Introduction. Minimal separating collections were introduced in [2, §8]; their knowledge is useful for the computation of the kernel of a cooperative game. In this note we determine an exact bound on the maximum number of sets which a minimal separating collection can have. The proof makes use of a result on finite graphs, which is proved in the next section.

2. Minimally ordered graphs. Throughout this paper we deal only with directed graphs. Our terminology is that of [1]. We recall some definitions that pertain to our work. A finite graph $G$ is a pair $(X, \Gamma)$, where $X$ is a finite set and $\Gamma$ is a multivalued function mapping $X$ into $X$, i.e., for each $x \in X$, $\Gamma(x)$ is a subset of $X$. Let $G = (X, \Gamma)$ be a finite graph. An element $x$ of $X$ is called a vertex. The number of vertices of $X$ is denoted by $n$. An ordered pair of vertices $(x, y)$ with $y \in \Gamma(x)$ is an arc. A path is a sequence of vertices $\mu = [x_1, \ldots, x_{k+1}]$ such that $x_{i+1} \in \Gamma(x_i)$ for $i = 1, \ldots, k$. Let $\mu = [x_1, \ldots, x_{k+1}]$ be a path. $\mu$ is a circuit if $x_1 = x_{k+1}$; if, in addition, $x_j \neq x_i$ for $i \neq j$, $1 \leq i, j \leq k$, then $\mu$ is an elementary circuit. $G$ is strongly connected if for every ordered pair of distinct vertices $x$ and $y$ there exists a path $\mu = [x, a_1, \ldots, a_{k-1}, y]$. A partial graph $G'$ of $G$ is a graph $(X, \Gamma')$, where $\Gamma'(x) \subseteq \Gamma(x)$ for every $x \in X$; $G'$ is a proper partial graph of $G$ if there exists a vertex $y$ such that $\Gamma'(y)$ is a proper subset of $\Gamma(y)$. Let $A$ be a subset of $X$. The subgraph of $G$ determined by $A$ is the graph $(A, \Gamma_A)$ where $\Gamma_A(a) = \Gamma(a) \cap A$, for all $a \in A$. An $s$-graph is a set $Y$ together with a collection $U$ of ordered pairs of members of $Y$; $U$ may contain the same ordered pair as many as $s$ times. Clearly a 1-graph is a graph. The shrinkage of $A$ is the $s$-graph obtained from $G$ by deleting the arcs of the subgraph $(A, \Gamma_A)$, and by identifying all the vertices of $A$. The number of members of $A$ is denoted by $|A|$. Let $x$ and $y$ be vertices of $G$. We write $x \leq y$ if $x = y$ or if there exists a path $\mu = [x, a_1, \ldots, a_{k-1}, y]$. The relation $\geq$ is the weak ordering associated with $G$. We write $x < y$ if $x \leq y$ but not $x \geq y$; we write $x \equiv y$ if $x \leq y$ and $y \leq x$. The relation $\equiv$ is the equivalence relation derived from $\leq$.

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Definition 2.1. A finite graph $G$ is minimally ordered if it has no proper partial graph which defines the same weak ordering on the vertices of $G$ as $G$.

Clearly a minimally ordered strongly connected graph is minimally connected [1, p. 123].

Lemma 2.2. A subgraph of a minimally ordered graph is minimally ordered.

The proof, which is straightforward, is omitted.

Lemma 2.3. Let $G$ be a minimally ordered graph and let $A \subseteq X$ determine a strongly connected subgraph; the shrinkage of $A$ leads to a minimally ordered graph.

The proof, which is similar to the proof of Theorem 1 [1, p. 123], is omitted.

Lemma 2.4. A minimally connected graph $G$ has at most $2(n-1)$ arcs.

Proof. The proof is by induction on $n$. For $n=1$ the lemma is true. Let $G$ be a minimally connected graph with $n$ vertices, $n \geq 2$. $G$ contains an elementary circuit $\mu$ with $l$ vertices, $l \geq 2$. By Theorem 1 [1, p. 123], the shrinkage of $\mu$ leads to a minimally connected graph $G'$. $G'$ has $n-l+1$ vertices. Since $n-l+1 < n$, $G'$ has, by assumption, no more than $2(n-l)$ arcs. By Theorem 2 [1, p. 124] $G$ has at most $2(n-l)+l=2n-l \leq 2n-2$ arcs.

Example 2.5. Let $X$ be a set with $n$ members, $n \geq 1$, and let $a \in X$. Define $\Gamma(a)=X-\{a\}$ and $\Gamma(x)=\{a\}$, for $x \in X-\{a\}$. $(X, \Gamma)$ is a minimally connected graph with $2(n-1)$ arcs.

Lemma 2.6. A minimally ordered graph without circuits has at most $f(n)$ arcs, where $f(n)=\frac{1}{4}n^2$ if $n$ is even, and $f(n)=\frac{1}{4}(n^2-1)$ if $n$ is odd.

Proof. The proof is by induction on $n$. For $n=1$ the lemma is true. Let $G=(X, \Gamma)$ be a minimally ordered graph without circuits with $n$ vertices. We shall assume that $n$ is even. The proof for odd $n$ is similar. Let $a$ be a minimal vertex of $G$, i.e., there exists no $x \in X$ such that $a>x$. Let $T=\{b \mid b>a \text{ and there exists no } c \text{ such that } b>c>a\}$. We remark that $x \in \Gamma(a)$ if and only if $x \in T$. If $|T| \leq \frac{1}{2}n$ let $X_1=X-\{a\}$. If $|T| > \frac{1}{2}n$ let $b \in T$ and $X_1=X-\{b\}$. We remark that the number of arcs incident into or out from $b$ is less than $\frac{1}{2}n$, since if $b_1 \in T$ then neither $b_1>b$ nor $b>b_1$. By Lemma 2.2 the subgraph determined by $X_1$ is minimally ordered; since it has no circuits it has, by assumption, at most $f(n-1)$ arcs. Hence $G$ has no more than $f(n-1)+\frac{1}{2}n=f(n)$ arcs.
Example 2.7. Let \( X \) be a set with \( n \) members, \( n \geq 1 \). Let \( A \subseteq X \) have \( \frac{1}{2}n \) members if \( n \) is even, and \( \frac{1}{2}(n+1) \) members if \( n \) is odd. Let \( \Gamma(a) = X - A \), for \( a \in A \). \( (X, \Gamma) \) is a minimally ordered graph without circuits which has \( f(n) \) arcs.

Lemma 2.8. A minimally ordered graph \( G = (X, \Gamma) \) has at most \( g(n) \) arcs, where \( g(n) = \max(2(n-1), f(n)) \).

Proof. Let \( X_1, \ldots, X_k \) be the equivalence classes determined by the equivalence relation derived from the weak ordering associated with \( G \). By Lemma 2.2, for \( i = 1, \ldots, k \), \( (X_i, \Gamma_{X_i}) \) is a minimally connected subgraph of \( G \). If \( k = n \) then \( G \) has no circuits. Thus if \( k = 1 \) or \( k = n \) the lemma is true by Lemmas 2.4 and 2.6. If \( 2 \leq k \leq n - 1 \) then, by Lemma 2.3, the shrinkage of \( X_1, \ldots, X_k \) leads to a minimally ordered graph \( G^* \). Clearly \( G^* \) has no circuits; hence it has no more than \( \frac{1}{2}k^2 \) arcs. By Lemma 2.4 each of the subgraphs \( (X_i, \Gamma_{X_i}), i = 1, \ldots, k \), has at most \( 2(|X_i| - 1) \) arcs. Thus \( G \) has no more than

\[
\frac{1}{2}k^2 + 2 \sum_{i=1}^{k} (|X_i| - 1) = \frac{1}{2}k^2 + 2(n - k) \leq g(n)
\]

arcs.

Corollary 2.9. Every finite graph \( G \) has a partial graph with no more than \( g(n) \) arcs which defines the same weak ordering on the vertices of \( G \) as \( G \).

3. Minimal separating collections. We recall some of the definitions of [2]. Let \( N \) be a finite nonempty set. Let \( \mathcal{D} \) be a collection of subsets of \( N \) and let \( i, j \in N, i \neq j \). \( i \) is separated by \( \mathcal{D} \) from \( j \) if there exists a set \( S \in \mathcal{D} \) such that \( i \in S \) and \( j \notin S \). \( \mathcal{D} \) is separating if for every pair \( i, j \in N \), \( i \) is separated from \( j \) by \( \mathcal{D} \) if and only if \( j \) is separated from \( i \) by \( \mathcal{D} \). A separating collection is minimal separating if it does not contain a proper subcollection which is separating. A collection \( \mathcal{D} \) is completely separating if for all pairs \( i, j \in N \), \( i \neq j \), \( i \) is separated from \( j \) and \( j \) is separated from \( i \) by \( \mathcal{D} \). A completely separating collection is minimal completely separating if it does not contain a proper subcollection which is completely separating. We remark that a completely separating collection which is minimal separating is a minimal completely separating collection.

Example 3.1. Let \( N \) be a finite nonempty set. The collections \( \mathcal{D}_1(N) = \{ \{i\} : i \in N \} \) and \( \mathcal{D}_2(N) = \{ N - \{i\} | i \in N \} \) are minimal separating and completely separating.

Example 3.2. Let \( N = \{1, \ldots, n\} \) be the set of the first \( n \) natural numbers. Let \( S_i = \{j | j \leq i\}, i = 1, \ldots, n - 1 \). The collection
\{S_1, \ldots, S_{n-1}, N-S_1, \ldots, N-S_{n-1}\} is minimal completely separating and has \(2(n-1)\) sets.

**Example 3.3.** Let \(N\) be a set with \(n\) members, \(n \geq 5\). Let \(A \subseteq N\) have \(\frac{1}{2}n\) members if \(n\) is even, and \(\frac{1}{2}(n-1)\) members if \(n\) is odd. The collection \(\{S \cup T \mid S \in \mathcal{D}_1(A), T \in \mathcal{D}_2(N-A)\}\) is minimal separating and completely separating and has \(f(n)\) sets (see Example 3.1, where \(\mathcal{D}_1\) and \(\mathcal{D}_2\) are defined, and Lemma 2.6 where \(f(n)\) is defined).

**Definition 3.4.** Let \(\mathcal{D}\) be a collection of subsets of a finite nonempty set \(N\). An ordered pair \((i, j)\) of members of \(N\) is a distinguished pair (with respect to \(\mathcal{D}\)) if there is exactly one set \(S \in \mathcal{D}\) such that \(i \in S\) and \(j \in S\).

**Lemma 3.5.** If \(\mathcal{D}\) is a minimal completely separating collection of subsets of a finite nonempty set \(N\), then for each \(S \in \mathcal{D}\) there exists a distinguished pair \((i, j)\) such that \(i \in S\) and \(j \in S\).

The proof, which is straightforward, is omitted.

**Theorem 3.6.** Let \(N\) be a finite nonempty set with \(n\) members. The maximum number of sets in a minimal completely separating collection of subsets of \(N\) is \(g(n)\) (see Lemma 2.8 where \(g(n)\) is defined).

**Proof.** Let \(\mathcal{D}\) be a minimal completely separating collection of subsets of \(N\). Define a graph \(G = (\mathcal{V}, \Gamma)\), where \(\Gamma\) is defined by: \(j \in \Gamma(i)\) if and only if \((i, j)\) is a distinguished pair (with respect to \(\mathcal{D}\)). By Corollary 2.9, \(G\) has a partial graph \(G' = (\mathcal{V}, \Gamma')\) which defines the same weak ordering as \(G\) and has no more than \(g(n)\) arcs. Let \(\mathcal{D}' \subseteq \mathcal{D}\) be defined by: \(\mathcal{D}' = \{S \mid S \in \mathcal{D}\ \text{and there exists a pair } (i, j) \text{ such that } j \in \Gamma'(i), i \in S \text{ and } j \in S\}\). \(\mathcal{D}'\) has at most \(g(n)\) sets. We shall show that \(\mathcal{D}' = \mathcal{D}\). Suppose, per absurdum, that there exists \(S \in \mathcal{D} - \mathcal{D}'\). By Lemma 3.5 there exists a distinguished pair \((i, j)\) such that \(i \in S\) and \(j \in S; j \geq i\) according to the ordering of \(G\). Since \(G'\) defines the same ordering, there exists a path \(\mu = [i, a_1, \ldots, a_{l-1}, j]\) in \(G'\). Let \(S_1, \ldots, S_i\) be sets in \(\mathcal{D}'\) such that \(i \in S_1, a_1 \in S_1, a_1 \in S_2, a_2 \in S_2, \ldots, a_{l-1} \in S_i, j \in S_i\). Since \(i \in S, S \neq S_1\) and \((i, a_1)\) is a distinguished pair, \(a_1 \in S\). Using induction we can show that all the vertices of \(\mu\) are in \(S\). Since \(j \in S\) we have a contradiction which shows that \(\mathcal{D}' = \mathcal{D}\). Hence the number of sets in \(\mathcal{D}\) is not greater than \(g(n)\). Examples 3.2 and 3.3 show that the bound \(g(n)\) is attained.

**Corollary 3.7.** Let \(N\) be a finite nonempty set with \(n\) members. The maximum number of sets in a minimal separating completely separating collection of subsets of \(N\) is \(f(n)\) for \(n \geq 7\), and is not greater than \(2(n-1)\) for \(n < 7\).
Proof. Theorem 3.6 and Example 3.3.

References


The Hebrew University of Jerusalem

IMBEDDING CLOSED RIEMANN SURFACES IN $C^n$

K. V. RAJESWARA RAO

I. Introduction. Let $R$ be a closed Riemann surface of genus $g$, $G$ a nonempty, open subset of $R$, and $A$ the set of all complex valued functions that are continuous on $R$ and holomorphic on $G$. With the usual pointwise operations $A$ is an algebra over the complex field. We consider the problem: how many functions in $A$ suffice to separate points of $R$?

Let $f$ be a nonconstant member of $A$. If the genus $g = 0$, Wermer [4] showed that there exist $f_1$ and $f_2$ in $A$ which, together with $f$ separate points of $R$; if $g = 1$, Arens [2] established the existence of $f_1$, $f_2$ and $f_3$ in $A$ which, together with $f$ separate points of $R$. In this note we shall present a modification of the Wermer-Arens argument to prove the following

Theorem. Let the genus $g$ be arbitrary. If $A$ contains nonconstant functions, then there exist four functions in $A$ which separate points of $R$ and which have no common branch points in $G$.

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II. Two lemmas. Let $\phi$ be a nonconstant member of order $n$ in the field $K$ of meromorphic functions on $R$. Let $w$ be a point of the extended plane which has $n$ distinct inverse images under $\phi$. Denote by $E(\phi, w)$ the finite set which is the union of $\phi^{-1}(w)$ and $\phi^{-1}(\phi(b))$ as $b$ ranges over all the branch points of $\phi$. For (fixed) $\phi$ and $\psi$ in $K$, let $S$

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