We consider ordinary graphs (finite, undirected, with no loops or multiple lines). A well-known concept in the theory of graphs is that of the (point) group $\Gamma(G)$ of a graph $G$, which is the group of all adjacency-preserving permutations of points of $G$. The elements of $\Gamma(G)$ are called (point-) automorphisms of $G$. In contrast with the notion of $\Gamma(G)$ is that of the line group $\Gamma'(G)$ of $G$ consisting of all adjacency-preserving permutations of lines of $G$. This group has been considered in [6]. The purpose of this paper is to treat another natural concept, called total group, and to prove that for any graph $G$ having more than one point, the total group of $G$ is isomorphic to the group of $G$ if and only if no component of $G$ is either a cycle or a complete graph.

1. Preliminaries. Denote the point set of $G$ by $V(G)$ and its line set by $X(G)$. Each member of $V(G) \cup X(G)$ will be called an element of $G$. We say two elements of $G$ are associated if they are either adjacent or incident. The group of all permutations of elements of $G$ which preserve association will be called the total group of $G$ and denoted by $\Gamma''(G)$.

The line graph [6] of a graph $G$, denoted by $L(G)$, is that graph whose point set is $X(G)$, and in which two points are adjacent if and only if they are adjacent in $G$. It is worth observing that $\Gamma'(G) \simeq \Gamma(L(G))$. The notion of total graphs introduced by one of the authors [1] is a convenient tool for our purposes. The total graph $T(G)$ of $G$ is that graph whose point set is $V(G) \cup X(G)$, and in which two points are adjacent if and only if they are associated in $G$. We should note that $\Gamma''(G)$ is isomorphic to $\Gamma(T(G))$. The graphs $G$ and $L(G)$ are disjoint subgraphs of $T(G)$. For illustration two graphs $G$ and $H$ are given in Figure 1 together with their line and total graphs. We observe that $\Gamma(G) \simeq \Gamma(T(G)) \simeq S_2$ (the symmetric group on two letters) while $\Gamma(H) \simeq S_2$ and $\Gamma(T(H)) \simeq S_3$.

The number of lines incident with a point $v$ of a graph $G$ is denoted by $\deg_G(v)$. For a point $v$ of $T(G)$ belonging to $V(G)$ we have $\deg_{T(G)}(v) = 2 \deg_G(v)$, and for a point $u$ of $T(G)$ belonging to $X(G) = V(L(G))$ we have $\deg_{T(G)}(u) = \deg_G(a) + \deg_G(b)$, where $a$ and $b$ are the points of $G$ incident with $u$. Thus if $\deg_{T(G)}(v)$ is maximal, then it is an even number.

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Definitions not given here may be found in [3] or [4].
If $a$ is a point of a graph $G$, then the subset of $V(G)$ consisting of the points adjacent with $a$ is called the neighborhood of $a$ in $G$ and is denoted by $N_G(a)$. The subgraph of $G$ generated by the points $a_1, a_2, \ldots, a_n$ of $G$ will be denoted by $\langle a_1, a_2, \ldots, a_n \rangle_G$. Finally, we denote by $K_p$ the complete graph of order $p$.

2. Results. We begin this section with

**Lemma 1.** Assume $G$ is any connected graph which is not a path (arc), a cycle, or a complete graph. Let $a_0$ be a point of $H = T(G)$ which has maximal degree $2d$. Then $a_0 \in V(G)$ if and only if the graph generated by $N_H(a_0)$ has exactly one $K_d$ as a subgraph.

**Proof.** The hypotheses imply that $d \geq 3$.

(i) Assume $a_0 \in V(G)$. The set $N_H(a_0)$ consists of $d$ points $a_i$ of $G$ and $d$ points $b_i$ of $L(G)$, $i = 1, 2, \ldots, d$. Since the $b_i$ are exactly those points of $L(G)$ which correspond to the lines $a_0a_i$ of $G$, which are all mutually adjacent in $G$, it follows that the $b_i$ are all mutually adjacent in $H$ and generate a $K_d$ in $H$. Now each $b_i$ is, by definition of total graphs, adjacent with exactly one of the $a_i$, $1 \leq i \leq d$, and therefore no subset of $N_H(a_0)$ which contains some $a_i$ together with some $b_i$ can generate a $K_d$. Nor can the $a_i$ themselves generate a $K_d$. For, otherwise $\langle a_0, a_1, \ldots, a_d \rangle_H = K_{d+1}$, and since $G$ is connected but not complete, there should exist at least one more point $a_{d+1}$ in $G$ adjacent
with some \( a_j \)—contradicting the maximality of \( d \). Thus the necessity of the condition is established.

(ii) Assume \( a_0 \in V(L(G)) \). We observe that \( N_H(a_0) \) consists of two points \( a_1 \) and \( a_2 \) of \( G \) and \( 2(d-1) \) points \( b_i \) of \( L(G) \). The latter points are those in \( L(G) \) which correspond to the lines of \( G \) incident with \( a_1 \) or with \( a_2 \); they are, by the maximality of \( d \), distributed equally between \( N_H(a_1) \) and \( N_H(a_2) \). Those of the \( b_i \) which are in \( N_H(a_1) \) are all mutually adjacent and generate a \( K_d \) together with \( a_1 \). Similarly \( a_2 \) and the remaining \( b_i \) generate a \( K_d \). This completes the proof of the lemma.

**Theorem 1.** Let \( G \) be a connected graph which is neither a cycle nor a complete graph. If \( H = T(G) \), then \( G \) is the only subgraph of \( H \) whose total graph is \( H \).

**Proof.** Let \( G' \) be any subgraph of \( H \) such that \( H = T(G') \). We shall show that \( G' = G \).

(i) Assume \( G \) is not a path. Take a point \( a_0 \) of \( G \) such that \( \deg_G(a_0) = d \) is maximal in \( G \). By Lemma 1 the point \( a_0 \) also belongs to \( G' \). We assert that \( N_G(a_0) = N_{G'}(a_0) \). This follows from the fact that the \( d \) points \( b_1, b_2, \ldots, b_d \) of \( N_H(a_0) - N_G(a_0) \) form the only \( K_d \) in \( N_H(a_0) \). By the proof of Lemma 1, the \( b_i \) which belong to \( L(G) \) should belong to \( L(G') \) as well. Let \( a_1, a_2, \ldots, a_d \) be the points of \( N_G(a_0) \). By the definition of total graphs two points of \( G \) can be adjacent in \( T(G) \) if and only if they are adjacent in \( G \). Thus \( A_1 = \{a_0, a_1, \ldots, a_d\}_H \) is contained in both \( G \) and \( G' \).

Now we proceed by induction. Assume that for \( k \geq 1 \) the points \( a_0, a_1, \ldots, a_{d+k-1} \) of \( H \) have been determined such that \( A_k = \{a_0, a_1, \ldots, a_{d+k-1}\}_H \) is contained in both \( G \) and \( G' \). Also assume that the subgraph \( L(A_k) \) of \( H \) is contained in \( L(G') \) as well as in \( L(G) \). If \( G \neq A_k \), then there exists at least one point \( a_{d+k} \) of \( G \) which is adjacent in \( G \) with some point \( a_j \) of \( A_k \). We show that \( a_{d+k} \) is also a point of \( G' \). If not, \( a_{d+k} \) belongs to \( L(G') \) and corresponds to a line of \( G' \) incident with \( a_j \), say \( a_ja \). By the induction hypotheses \( c \) is not a point of \( A_k \). There is at least one line in \( A_k \) incident with \( a_j \), say \( a_ja \); we denote the point of \( L(A_k) \) corresponding to this line by \( b \). The points \( b \) and \( a_{d+k} \) have to be adjacent in \( T(G') \); but this implies the contradiction that in \( T(G) \) the line \( b \) corresponds to the two lines \( a_ja \) and \( a_ja_{d+k} \). Thus \( A_{k+1} = \{a_0, a_1, \ldots, a_{d+k}\}_H \) is a subgraph of both \( G \) and \( G' \).

Let \( a_i \) be any point of \( A_k \) adjacent with \( a_{d+k} \). Denote by \( b' \) the point of \( L(G') \) corresponding to \( a_{d+k}a_i \). If \( b' \) is not in \( L(G) \), then it is a point of \( G \). But then the proof, given above, for the assertion that \( A_{d+k} \) belongs to \( G' \) is applicable to \( b' \) and leads to a contradiction. Thus \( b' \)
is the point of $L(G)$ corresponding to the line $a_{d+k}a_i$. Hence $L(A_{k+1})$ is a subgraph of both $L(G)$ and $L(G')$. This completes the induction.

If $p$ is the order of $G$, then $A_{p-d}$ implies that $G$ is a subgraph of $G'$ and that $L(G)$ is a subgraph of $L(G')$. This proves at once that $G = G'$.

(ii) Let $G$ be a path of order $p$. Since $G$ is not complete we have $p \geq 3$. The graph $H$ has exactly two points $a_1$ and $a_p$ of degree 2 which are the end points of the path $G$. Since the degree of no point of $H$ exceeds 4, the degree of no point of $G'$ can exceed 2. Thus $G'$ is a path with end points $a_1$ and $a_p$. Observing that the order of $G'$ should also be $p$, and that $G$ is the only path of order $p$ in $H$ with end points $a_1$ and $a_p$, it follows that $G' = G$. This completes the proof of Theorem 1.

Considering disconnected graphs, we note that if $G$ has $n$ components $G_i$, then $T(G)$ has $n$ components $T(G_i)$, $i = 1, 2, \ldots, n$, and vice versa. Thus we have

**Corollary 1.** Let $G$ be a graph none of whose components is a cycle or a complete graph. If $T(G) = T(G')$ for any subgraph $G'$ of $T(G)$, then $G' = G$.

The following theorem is of interest in itself; it is the analog of the theorem of Whitney [6] for line-graphs which states that given any two connected graphs $G_1$ and $G_2$ other than $K_3$ and $K_{1,3}$ we have $L(G_1) \simeq L(G_2)$ if and only if $G_1 \simeq G_2$. (The symbol $\simeq$ indicates isomorphism.)

**Theorem 2.** Let $G_1$ and $G_2$ be two graphs. Then $T(G_1) \simeq T(G_2)$ if and only if $G_1 \simeq G_2$.

**Proof.** It suffices to prove the result for connected graphs. The isomorphism of $T(G_1)$ and $T(G_2)$ follows from that of $G_1$ and $G_2$ by definition. Thus we show the converse. In view of Theorem 1 we can confine ourselves to the case in which $G_1$ is either a cycle or a complete graph. Since a graph is regular of degree $d$ and order $p$ if and only if its total graph is regular of degree $2d$ and order $p(1+d/2)$, it follows that in both cases $G_2$ is regular and has the same degree and order as $G_1$. This implies, in both cases, that $G_1 \simeq G_2$.

Next we state our main result.

**Theorem 3.** For any graph $G$ other than a single point the two groups $\Gamma''(G)$ and $\Gamma(G)$ are isomorphic if and only if no component of $G$ is either a complete graph or a cycle.

**Proof.** Each automorphism $f$ of $G$ can be extended to an automorphism $f'$ of $T(G)$ as follows: if $a$ is a point of $V(T(G)) - V(G)$, then
it corresponds to some line $bc$ of $G$, and we let $f'(a) = a'$, where $a'$ is the point of $T(G)$ corresponding to $f(b)f(c)$. The function $f'$ is indeed an automorphism of $T(G)$, because (1) if $a \in V(T(G)) - V(G)$, then it is adjacent with exactly two points $a_1$ and $a_2$ of $G$; by definition of $f'$ the point $f'(a)$ is adjacent to the two points $f'(a_1)$ and $f'(a_2)$, and (2) if $a_1$ and $a_2$ are two adjacent points of $V(T(G)) - V(G)$, then the adjacency of $f'(a_1)$ and $f'(a_2)$ follows from the fact that the restriction of $f'$ to $L(G)$ is the automorphism induced [5] by $f$.

If $f_1$ and $f_2$ are any two automorphisms of $G$, we observe that $(f_1f_2)' = f_1'f_2'$. Furthermore, if $f_1$ and $f_2$ are distinct, then so are $f_1'$ and $f_2'$, so that $\Gamma(T(G))$ (which is the same as $\Gamma''(G)$) contains an isomorphic image of $\Gamma(G)$.

Let $g \in \Gamma(T(G))$. If we denote by $G'$ the image under $g$ of the subgraph $G$ of $T(G)$, then the image of the subgraph $L(G)$ is a subgraph which is necessarily $L(G')$. Also if $v$ is a point of $L(G)$ adjacent with a point $u$ of $G$, then $g(v)$ is a point of $L(G')$ adjacent with the point $g(u)$ of $G'$. Hence $T(G) = T(G')$.

If $G$ has no complete graph or cycle as a component, then $G = G'$ by Corollary 1 of Theorem 1. Thus every automorphism of $T(G)$ is an extension of some automorphism of $G$, implying $\Gamma''(G) = \Gamma(G)$.

If the above subgraphs $G$ and $G'$ are distinct, then for no automorphism $f$ of $G$ we have $g = f'$. So $\Gamma''(G)$ is not isomorphic with $\Gamma(G)$. To complete the proof of the theorem, it suffices to show that if $G$ is either a cycle or a complete graph other than $K_1$, then $T(G)$ contains a subgraph $G'$, $G' \neq G$, with $T(G) = T(G')$.

(i) If $G$ is a cycle, then the subgraph $L(G)$ of $T(G)$ is isomorphic to $G$ and we have $T(L(G)) = T(G)$.

(ii) Now let $G = K_p$, $p > 1$, $H = T(G)$, and let $a_0$ be a point of the subgraph $G$ of $H$. Denote the points of $N_G(a_0)$ by $a_1, a_2, \ldots, a_{p-1}$. For each $i$, $1 \leq i \leq p-1$, let $b_i$ denote the point of the subgraph $L(G)$ of $H$ which corresponds to the line $a_0a_i$. We shall prove that $T(G') = H$, where $G' = \langle a_0, b_1, \ldots, b_{p-1} \rangle_H$. Let $f$ be the permutation of $V(H)$ defined by: $f(a_i) = b_i, f(b_i) = a_i$ for $1 \leq i \leq p-1$, and $f(v) = v$ otherwise. The function $f$ is an automorphism of $H$. In fact, for $1 \leq i \leq p-1$, the $a_i$ are mutually adjacent and so are the $b_i$; hence it would suffice to show that if $v, v \neq a_0$, is a point of $H$ different from the $a_i$ and $b_i$, $1 \leq i \leq p-1$, which is adjacent with some $a_k$, then $v$ is adjacent with $b_k$ and conversely. Since $v$ and $b_k$ are both points of $L(G)$, they are adjacent if and only if they correspond to adjacent lines of $G$. This occurs if and only if $a_0a_k$ and the line of $G$ corresponding to $v$, say $a_ra_s$, $1 \leq r < s \leq p-1$, are adjacent, which is true if and only if either
$r = k$ or $s = k$. Since the only $a_i$ adjacent with $v$ are $a_r$ and $a_s$, our assertion follows.

**Remark.** Let $G$ be either a cycle or a complete graph. The subgraphs of $T(G)$ whose total graphs coincide with $T(G)$ have been enumerated elsewhere [2]. Here we treat these cases briefly. We first observe that if $G$ and $G'$ are any two subgraphs of $H$ such that $H = T(G) = T(G')$, then every isomorphism of $G$ onto $G'$ gives rise to an automorphism of $H$.

(i) For $G = K_3$, $T(G)$ is the total graph of each of its eight triangles; $\Gamma''(G)$ has order 48.

(ii) For a cycle $G$ of order $p$, $p > 3$, the only other subgraph of $T(G)$ whose total graph is $T(G)$ is $L(G)$; $\Gamma''(G)$ has order $4p$.

(iii) For a complete graph $G$ of order $p$, $p > 3$, there are exactly $p$ other subgraphs $G_i$ of $T(G)$ with $T(G_i) = T(G)$, $i = 1, 2, \ldots, p$. Thus the order of $\Gamma''(G)$ is $(p+1)!$.

We conclude the paper with the observation that for a connected graph $G$, $\Gamma(G) \cong \Gamma'(G) \cong \Gamma''(G)$ if and only if $G$ is not (1) a complete graph, (2) a cycle, or (3) the graph $K_4$ minus one or two lines. This follows from our results together with [6].

**Bibliography**


**Pahlavi University, Shiraz, Iran**