A NOTE ON THE KOSZUL COMPLEX

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It is well known that the Koszul complex $R_{x_1, \ldots, x_s}$ of an ideal $I = (x_1, \ldots, x_s)$ over a commutative ring $R$ possesses the structure of an exterior algebra (see [3, p. 189]). Use of this structure is avoided in [1] and [5], but one can give easier proofs of some of the results by using it, e.g. the fact that the ideal $I$ annihilates the homology of the complex (see [3]). This note gives an easy proof of a more general result than in paragraph 2 of [2] or of a remark in [5, p. IV-12]. The extra generality is necessary in the theory of multiplicities and Grothendieck groups developed in [4].

Our notation is that of [1] and [2]. We remark that the complex $E_{x_1, \ldots, x_s} = R_{x_1, \ldots, x_s} \otimes_R E$ has the structure of a graded left module over the exterior algebra $R_{x_1, \ldots, x_s}$. The exterior product will be denoted $\wedge$.

**Proposition.** Let $R$ be a commutative Noetherian ring with unit. Let $I = (x_1, \ldots, x_s)$. Let $E$ be a finitely generated module over $R$. Let $C$ be the complex $E_{x_1, \ldots, x_s}$. Let $C^{(k)}$ be the subcomplex of $C$ given by

$$0 \rightarrow I^k C_k \rightarrow I^{k+1} C_{k-1} \rightarrow \cdots \rightarrow I^{k+s} C_0 \rightarrow 0.$$ 

Then there exists an integer $n$ such that $C^{(k)}$ is acyclic for all $k > n$. Note that no assumptions concerning finite length are made.

**Proof.** Let $Z_i$ be the cycles in $C_i$. Then $Z_i \cap I^{k+s-i} C_i$ are the cycles in $I^{k+s-i} C_i$. By the Artin-Rees lemma we may choose $n$ such that for $k$ larger than $n$ $Z_i \cap I^{k+s-i} C_i = I(Z_i \cap I^{k+s-i-1} C_i)$, and we may certainly do this uniformly for all $i$. Let $z$ be a cycle in $I^{k+s-i} C_i$. By the above, $z = \sum_{i=1}^s x_i a_i$ where $a_i \in Z_i \cap I^{k+s-i-1} C_i$.

Let $u_i$ be the element of $(R_{x_1, \ldots, x_s})_1$ such that $d(u_i) = x_i$ when $x_i$ is considered as an element of $(R_{x_1, \ldots, x_s})_0$. Let $w \sum_{i=1}^s u_i \wedge a_i$. $w$ is in $I^{k+s-i-1} C_{i+1}$.

$$d(w) = \sum_{i=1}^s (d(u_i) \wedge a_i - u_i \wedge d(a_i)) = \sum_{i=1}^s (x_i a_i - 0) = z.$$

Q.E.D.

**References**


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THE DISTANCE FROM $U(z) \cdot H^p$ TO 1

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If $U(z)$ is an inner function, then the set $U(z) \cdot H^p$ of all $H^p$ multiples of $U(z)$ forms a closed subspace of $H^p$. In this note we compute the $H^p$ distance between the constant function 1 and this closed subspace. It is of course well known that this distance is 0 if and only if $U(z)$ is a constant, i.e. if and only if $|U(0)| = 1$. We will prove

**Theorem.** \( \text{dist} (1, U(z) \cdot H^p) = (1 - |U(0)|^2)^{1/p}, \quad p \geq 1. \)

**Proof.** With

\[
f_p(z) = \frac{1 - (1 - U(z)U(0))^{2/p}}{U(z)}
\]

we have

\[
\|1 - U(z)f_p(z)\| = \left( \frac{1}{2\pi} \int_0^{2\pi} |1 - U(z)U(0)|^2 d\theta \right)^{1/p}
\]

\[
= \left( \frac{1}{2\pi} \int_0^{2\pi} |U(z) - U(0)|^2 d\theta \right)^{1/p} = (1 - |U(0)|^2)^{1/p},
\]

so that this distance is surely \( \leq (1 - |U(0)|^2)^{1/p} \) and we need only show that $f_p(z)$ is the closest function to 1, i.e. that, for all $f(z) \in H^p$,

\[
\|1 - U(z)f(z)\|_p \geq (1 - |U(0)|^2)^{1/p}.
\]

Consider

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